

# Explicit Coupling Argument for Nonuniformly Hyperbolic Transformations

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## Abstract

The transfer operator corresponding to a uniformly expanding map enjoys good spectral properties. Here it is verified that coupling yields explicit estimates that depend continuously on the expansion and distortion constants of the map.

For nonuniformly expanding maps with a uniformly expanding induced map, we obtain explicit estimates for mixing rates (exponential, stretched exponential, polynomial) that again depend continuously on the constants for the induced map together with data associated to the inducing time.

Finally, for nonuniformly hyperbolic transformations, we obtain the corresponding estimates for rates of decay of correlations.

## 1 Introduction

It is well-known that the transfer operator associated to a uniformly expanding map enjoys good spectral properties. In particular, there are numerous methods for proving exponential decay of correlations for uniformly expanding maps, see for example [1, 9, 24, 25, 27].

Often, statistical properties of nonuniformly expanding systems are studied by inducing to a uniformly expanding one. Young [30, 31] obtained results on decay of correlations for large classes of such nonuniformly expanding maps, as well as nonuniformly hyperbolic transformations. The rate of decay is related to the tails of the inducing time, with special emphasis placed on exponential tails and polynomial tails. Stretched exponential decay rates (amongst others) were obtained in Maume-Deschamps [21]. The resulting decay rates have the form  $O(e^{-cn^\gamma})$  or  $O(n^{-\beta})$  where  $\gamma \in (0, 1]$  and  $\beta > 0$  are given explicitly, but the constant  $c > 0$  and the implied constants are not. An improved estimate of Gouëzel [14] in the stretched exponential

case  $\gamma \in (0, 1)$  determines also  $c$ , but the implied constant remains nonexplicit as does the constant  $c$  in the exponential case  $\gamma = 1$ .

In this paper, we use an explicit coupling argument to obtain mixing rates with uniform control on the various constants. The main novelty in our results lies in the nonuniformly expanding/hyperbolic setting. However, even for uniformly expanding maps, we expect that our results have numerous applications, see for example [18, 19].

Related results using the coupling method for uniformly expanding maps can be found in both simpler and more complicated situations (usually in low dimensions) in recent papers, for example [12, 28]. See also [20] for an approach using Birkhoff cones for one-dimensional maps. None of these results are formulated in such a way that they can be cited in [18, 19]. In this paper, we work in a general metric space and present a much shorter and more elementary proof than was previously written down. The results then feed into the more complicated argument required in the nonuniformly expanding/hyperbolic setting.

**Remark 1.1** Zweimüller [32] uses a coupling argument for uniform expanding maps defined on a general compact metric space; compactness being used to obtain an invariant density via an Arzelà-Ascoli argument. The proof below of Proposition 2.5 shows how to bypass this, so that compactness of the metric space is not required. (For an alternative argument, see [2] or [1, Lemma 4.4.1].) In all other respects, the treatment in [32] seems as straightforward as the one presented here and holds in a more general context (we assume full branches which simplifies matters somewhat and is sufficient for our purposes in [18, 19]). Indeed, it is implicit in [32] that the desired explicit control on various constants (for uniformly expanding maps) is obtained. For ease of reference, we make explicit this control of constants.

**Remark 1.2** Keller & Liverani [17] considered continuous families of uniformly expanding maps and developed a perturbative theory that gives uniform estimates on the spectra of the associated transfer operators. This idea was used by [11] in the situation of dispersing billiards. However, inducing from continuous families of nonuniformly expanding maps to families of uniformly expanding maps may fail to preserve any useful notion of continuous dependence. In particular, the examples in [18, Section 5] and in [19] do not satisfy the hypotheses of [11, 17].

In this paper, we do not assume any continuous dependence on parameters. Instead, we work with a fixed uniformly expanding map  $F$ , and give explicit estimates on the associated transfer operator that depend continuously on the expansion and distortion estimates of  $F$ .

Even for nonuniformly expanding/hyperbolic dynamical systems, none of the results in this paper are particularly surprising. Nevertheless, the results go far beyond those previously available. Some examples are listed at the end of Section 2.2. In the case of smooth unimodal maps there are previous results [8, Theorem 1.3] showing exponential decay of correlations up to a finite period with uniform exponent (uni-

formity of the implied constant is not claimed in [8]). Here we obtain a similar result with uniform exponent and uniform implied constant. In the case of families of Viana maps [29] which are known to have stretched exponential decay of correlations [15], we obtain for the first time uniform estimates on the constants  $C$ ,  $c$ ,  $\gamma$  in the stretched exponential decay rate  $Ce^{-cn^\gamma}$ .

Our main results are stated in Section 2 and proved for uniformly expanding, nonuniformly expanding, and nonuniformly hyperbolic, transformations in Sections 3, 4 and 5 respectively.

## 2 Statement of the main results

In this section, we state our main results for uniformly expanding maps (Subsection 2.1), nonuniformly expanding maps (Subsection 2.2), and nonuniformly hyperbolic transformations (Subsection 2.3).

### 2.1 Uniformly expanding maps

Let  $(Y, m)$  be a probability space, and  $F : Y \rightarrow Y$  be a nonsingular transformation. Let  $d$  be a metric on  $Y$  such that  $\text{diam } Y \leq 1$ .

Suppose that  $\alpha$  is an at most countable measurable partition of  $Y$ , and that  $F$  restricts to a measure-theoretic bijection from  $a$  onto  $Y$  for each  $a \in \alpha$ .

Let  $\zeta = \frac{dm}{dm \circ F}$  be the inverse Jacobian of  $F$  with respect to  $m$ . Assume that there are constants  $\lambda > 1$ ,  $K > 0$  and  $\eta \in (0, 1]$  such that for  $x, y$  in the same partition element

$$d(Fx, Fy) \geq \lambda d(x, y) \quad \text{and} \quad |\log \zeta(x) - \log \zeta(y)| \leq Kd(Fx, Fy)^\eta. \quad (2.1)$$

Let  $P_m : L^1(Y) \rightarrow L^1(Y)$  be the transfer operator corresponding to  $F$  and  $m$ , so  $\int_Y P_m \phi \psi dm = \int_Y \phi \psi \circ F dm$  for all  $\phi \in L^1$  and  $\psi \in L^\infty$ . Then  $P_m \phi$  is given explicitly by

$$(P_m \phi)(y) = \sum_{a \in \alpha} \zeta(y_a) \phi(y_a),$$

where  $y_a$  is the unique preimage of  $y$  under  $F$  lying in  $a$ .

Given  $\phi : Y \rightarrow \mathbb{R}$ , define

$$|\phi|_\eta = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{d(x, y)^\eta} \quad \text{and} \quad \|\phi\|_\eta = |\phi|_\infty + |\phi|_\eta.$$

Let  $C^\eta$  denote the Banach space of observables  $\phi : Y \rightarrow \mathbb{R}$  such that  $\|\phi\|_\eta < \infty$ .

It is well-known that there exist constants  $C > 0$ ,  $\gamma \in (0, 1)$ , such that  $\|P_m^n \phi\|_\eta \leq C\gamma^n \|\phi\|_\eta$  for all  $\phi \in C^\eta$  with  $\int_Y \phi dm = 0$  and all  $n \geq 1$ . Our main result is:

**Theorem 2.1** *There exist constants  $C > 0$ ,  $\gamma \in (0, 1)$  depending continuously on  $\lambda$ ,  $K$  and  $\eta$ , such that*

$$\|P_m^n \phi\|_\eta \leq C\gamma^n |\phi|_\eta,$$

*for all  $\phi \in C^n$  with  $\int_Y \phi \, dm = 0$ , and all  $n \geq 1$ .*

**Remark 2.2** For example, take  $R = 2K/(1 - \lambda^{-\eta})$  and  $\xi = \frac{1}{2}e^{-R}(1 - \lambda^{-\eta})$ . Then Theorem 2.1 holds with  $C = 4e^R(1 + R)$  and  $\gamma = 1 - \xi$ .

Next, let  $\mathcal{M}$  be the collection of probability measures on  $Y$  that are equivalent to  $m$  and satisfy  $L_\mu < \infty$  where  $L_\mu = |\log \frac{d\mu}{dm}|_\eta$ . Given  $\mu \in \mathcal{M}$ , define  $\zeta_\mu = \frac{d\mu}{d\mu \circ F}$  and let  $P_\mu$  be the corresponding transfer operator.

**Proposition 2.3** *For all  $x, y$  in the same partition element,*

$$|\log \zeta_\mu(x) - \log \zeta_\mu(y)| \leq K_\mu d(Fx, Fy)^\eta,$$

*where  $K_\mu = K + (\lambda^{-\eta} + 1)L_\mu$ .*

**Proof** Note that  $\log \zeta_\mu = \log \zeta + h - h \circ F$  where  $h = \log \frac{d\mu}{dm}$ . Hence  $|\log \zeta_\mu(x) - \log \zeta_\mu(y)| \leq |\log \zeta(x) - \log \zeta(y)| + |h|_\eta d(x, y)^\eta + |h|_\eta d(Fx, Fy)^\eta \leq (K + L_\mu \lambda^{-\eta} + L_\mu) d(Fx, Fy)^\eta$ . ■

In other words, the hypotheses of Theorem 2.1 are satisfied with  $m$  and  $K$  replaced by  $\mu$  and  $K_\mu$ . Hence, we obtain:

**Corollary 2.4** *Let  $\mu \in \mathcal{M}$ . There exist constants  $C > 0$ ,  $\gamma \in (0, 1)$  depending continuously on  $\lambda$ ,  $K_\mu$  and  $\eta$ , such that*

$$\|P_\mu^n \phi\|_\eta \leq C\gamma^n |\phi|_\eta,$$

*for all  $\phi \in C^n$  with  $\int_Y \phi \, d\mu = 0$ , and all  $n \geq 1$ .* ■

Of special interest is the case where  $\mu$  is the unique absolutely continuous  $F$ -invariant probability measure. For this special case, we prove:

**Proposition 2.5** *The invariant probability measure  $\mu$  lies in  $\mathcal{M}$ , and there is a constant  $R$  depending continuously on  $\lambda$  and  $K$  (chosen as in Remark 2.2 say) such that*

$$e^{-R} \leq \frac{d\mu}{dm} \leq e^R, \quad \left| \log \frac{d\mu}{dm} \right|_\eta \leq R.$$

*In particular, the constants  $C$  and  $\gamma$  in Corollary 2.4 depend continuously on  $\lambda$ ,  $K$  and  $\eta$ .*

**Remark 2.6** A standard extension of these results is to treat observables  $\phi : Y \rightarrow \mathbb{R}$  that are piecewise Hölder (relative to the partition  $\alpha$ ) and possibly unbounded. Provided  $P_m \phi \in C^\alpha$ , our results go through unchanged (with obvious modifications to the constant  $C$ ). For instances of this extension, we refer to [22, Lemma 2.2] or [18, Proposition 4.7].

## 2.2 Nonuniformly expanding maps

Let  $F : Y \rightarrow Y$  be a uniformly expanding map with probability measure  $m$  (not necessarily invariant), constants  $\lambda$ ,  $K$  and  $\eta$ , and partition  $\alpha$ , as in Subsection 2.1. Let  $\tau : Y \rightarrow \mathbb{Z}^+$  be an integrable function that is constant on partition elements. Define the Young tower [31]

$$\Delta = \{(y, \ell) \in Y \times \mathbb{Z} : 0 \leq \ell \leq \tau(y) - 1\}$$

and  $f : \Delta \rightarrow \Delta$ ,

$$f(y, \ell) = \begin{cases} (y, \ell + 1), & \ell \leq \tau(y) - 2, \\ (Fy, 0), & \ell = \tau(y) - 1. \end{cases}$$

Let  $\bar{\tau} = \int_Y \tau dm$ . Let  $m_\Delta$  be the probability measure on  $\Delta$  given by  $m_\Delta(A \times \{\ell\}) = \bar{\tau}^{-1}m(A)$  for all  $\ell \geq 0$  and measurable  $A \subset \{y \in Y : \tau(y) \geq \ell + 1\}$ .

Let  $d_\Delta$  be the metric on  $\Delta$  given by

$$d_\Delta((y, \ell), (y', \ell')) = \begin{cases} 1, & \ell \neq \ell' \\ d(y, y'), & \ell = \ell'. \end{cases}$$

Given  $\phi : \Delta \rightarrow \mathbb{R}$ , define  $|\phi|_\eta = \sup_{x, y \in \Delta} \frac{|\phi(x) - \phi(y)|}{d_\Delta(x, y)^\eta}$  and  $\|\phi\|_\eta = |\phi|_\eta + |\phi|_\infty$ .

Let  $L : L^1(\Delta) \rightarrow L^1(\Delta)$  denote the transfer operator corresponding to  $f$  and  $m_\Delta$ , so  $\int_\Delta L\phi \psi dm_\Delta = \int_\Delta \psi \circ f d\mu$  for all  $\phi \in L^1$ ,  $\psi \in L^\infty$ .

When the measure  $m$  on  $Y$  is  $F$ -invariant,  $m_\Delta$  is an ergodic  $f$ -invariant probability measure on  $\Delta$  and  $m_\Delta$  is mixing under  $f$  if and only if  $\gcd\{\tau(a) : a \in \alpha\} = 1$ . Accordingly, we say that the tower  $f : \Delta \rightarrow \Delta$  is *mixing* if  $\gcd\{\tau(a) : a \in \alpha\} = 1$ , and *nonmixing* otherwise, even though we do not assume that  $m_\Delta$  is  $f$ -invariant.

**Mixing Young towers** In the mixing case, there exist  $\delta > 0$  and a finite set of positive integers  $\{I_k\}$  with  $\gcd\{I_k\} = 1$  such that  $m(\{y \in Y : \tau(y) = I_k\}) \geq \delta$ .

**Theorem 2.7** *Let  $\phi : \Delta \rightarrow \mathbb{R}$  be an observable with  $\|\phi\|_\eta < \infty$  and  $\int_\Delta \phi dm_\Delta = 0$ .*

- *Suppose that  $m(\tau \geq n) \leq C_\tau n^{-\beta}$  for some  $\beta > 1$  and all  $n > 0$ . Then there exists a constant  $C > 0$  depending continuously on  $\lambda$ ,  $K$ ,  $\eta$ ,  $\max\{I_k\}$ ,  $\delta$ ,  $\beta$  and  $C_\tau$ , such that for all  $n \geq 0$*

$$\int_\Delta |L^n \phi| dm_\Delta \leq C \|\phi\|_\eta n^{-(\beta-1)}.$$

- *Suppose that  $m(\tau \geq n) \leq C_\tau e^{-An^\gamma}$  for some  $A > 0$ ,  $0 < \gamma \leq 1$  and all  $n > 0$ . Then there exist constants  $B > 0$  and  $C > 0$  depending continuously on  $\lambda$ ,  $K$ ,  $\eta$ ,  $\max\{I_k\}$ ,  $\delta$ ,  $A$ ,  $\gamma$  and  $C_\tau$ , such that for all  $n \geq 0$*

$$\int_\Delta |L^n \phi| dm_\Delta \leq C \|\phi\|_\eta e^{-Bn^\gamma}.$$

**Nonmixing Young towers** In the nonmixing case, define

$$d = \gcd\{j \geq 1 : m(\{y \in Y : \tau(y) = j\}) > 0\} \geq 2.$$

There exist  $\delta > 0$  and a finite set of positive integers  $\{I_k\}$  with  $\gcd\{I_k\} = d$  such that  $m(\{y \in Y : \tau(y) = I_k\}) \geq \delta$ .

**Theorem 2.8** *Let  $\phi : \Delta \rightarrow \mathbb{R}$  be an observable with  $\|\phi\|_\eta < \infty$  and  $\int_\Delta \phi dm_\Delta = 0$ . Then Theorem 2.7 holds with  $\int_\Delta |L^n \phi| dm_\Delta$  replaced by  $\int_\Delta |\sum_{k=0}^{d-1} L^{nd+k} \phi| dm_\Delta$ .*

Theorem 2.8 has the following equivalent reformulation which gives uniform mixing rates up to a cycle of length  $d$ . We state the reformulation for the case of (stretched) exponential mixing. The polynomial mixing case goes the same way.

Write  $\Delta = E_1 \cup \dots \cup E_d$  where  $f(E_j) = E_{j+1 \bmod d}$  and  $f^d : E_j \rightarrow E_j$  is a mixing tower for  $j = 1, \dots, d$ .

**Corollary 2.9** *Suppose that we are in the situation of Theorem 2.8 and that  $m(\tau \geq n) \leq C_\tau e^{-An^\gamma}$  as in the second part of Theorem 2.7. Fix  $j = 1, \dots, d$ . Then there exist uniform constants  $B, C > 0$  as in Theorem 2.7 such that*

$$\left| \int_\Delta \phi \psi \circ f^{nd} dm_\Delta - \int_\Delta \phi dm_\Delta \int_\Delta \psi dm_\Delta \right| \leq C \|\phi\|_\eta |\psi|_\infty e^{-Bn^\gamma},$$

for all  $n \geq 1$  and all  $\phi, \psi \in L^\infty$  supported in  $E_j$  with  $\|\phi\|_\eta < \infty$ .

**Examples** In [18, 19], we verified for specific families of nonuniformly expanding maps that the corresponding induced maps  $F$  are uniformly expanding, as in Subsection 2.1, with uniform constants  $\lambda, K, \eta$ . A key ingredient in this verification is the work of [3, 5, 7, 13] on strong statistical stability (where the density of the invariant measure varies continuously in  $L^1$ ). It follows from this abstract framework (specifically condition (U1) in [7]) that the data  $d = \gcd\{I_k\} \geq 1$  and  $\delta > 0$  associated with the inducing time  $\tau$  varies continuously in the mixing case and upper semicontinuously in general (so  $d$  can decrease under small perturbations but cannot increase). Hence for the examples in [18, 19], uniform estimates on decay of correlations follow immediately from Theorems 2.7 and 2.8.

Specifically, we obtain uniform polynomial decay of correlations for intermittent maps [19, Example 4.9], uniform exponential decay of correlations (up to a finite cycle) for smooth unimodal and multimodal maps satisfying the Collet-Eckmann condition [19, Example 4.10], and uniform stretched exponential decay of correlations for Viana maps [19, Example 4.11].

## 2.3 Nonuniformly hyperbolic transformations

Let  $T : M \rightarrow M$  be a diffeomorphism (possibly with singularities) defined on a Riemannian manifold  $(M, d)$ . Fix a subset  $Y \subset M$ . It is assumed that there is

a “product structure”: namely a family of “stable disks”  $\{W^s\}$  that are disjoint and cover  $Y$ , and a family of “unstable disks”  $\{W^u\}$  that are disjoint and cover  $Y$ . Each stable disk intersects each unstable disk in precisely one point. The stable and unstable disks containing  $y$  are labelled  $W^s(y)$  and  $W^u(y)$ .

Suppose that there is a partition  $\{Y_j\}$  of  $Y$  and integers  $\tau(j) \geq 1$  with  $\gcd\{\tau(j)\} = 1$  such that  $T^{\tau(j)}(W^s(y)) \subset W^s(T^{\tau(j)}y)$  for all  $y \in Y_j$ . Define the return time function  $\tau : Y \rightarrow \mathbb{Z}^+$  by  $\tau|_{Y_j} = \tau(j)$  and the induced map  $F : Y \rightarrow Y$  by  $F(y) = T^{\tau(y)}(y)$ .

Let  $s$  denote the *separation time* with respect to the map  $F : Y \rightarrow Y$ . That is, if  $y, z \in Y$ , then  $s(y, z)$  is the least integer  $n \geq 0$  such that  $F^n y, F^n z$  lie in distinct partition elements of  $Y$ .

(P1) There exist constants  $K_0 \geq 1$ ,  $\rho_0 \in (0, 1)$  such that

- (i) If  $z \in W^s(y)$ , then  $d(F^n y, F^n z) \leq K_0 \rho_0^n$ ,
- (ii) If  $z \in W^u(y)$ , then  $d(F^n y, F^n z) \leq K_0 \rho_0^{s(y, z) - n}$ ,
- (iii) If  $y, z \in Y$ , then  $d(T^j y, T^j z) \leq K_0 d(y, z)$  for all  $0 \leq j < \min\{\tau(y), \tau(z)\}$ .

Let  $\bar{Y} = Y / \sim$  where  $y \sim z$  if  $y \in W^s(z)$  and define the partition  $\{\bar{Y}_j\}$  of  $\bar{Y}$ . We obtain a well-defined return time function  $\tau : \bar{Y} \rightarrow \mathbb{Z}^+$  and induced map  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$ . Suppose that the map  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  and partition  $\alpha = \{\bar{Y}_j\}$  separate points in  $\bar{Y}$ , and let  $s$  denote also the separation time on  $\bar{Y}$ . Fix  $\theta \in (0, 1)$ . Then  $d_\theta(y, z) = \theta^{s(y, z)}$  defines a metric on  $\bar{Y}$ . Suppose further that  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  is a uniformly expanding map in the sense of Subsection 2.1 on the metric space  $(\bar{Y}, d_\theta)$ , with partition  $\alpha$  and constants  $\lambda = 1/\theta > 1$ ,  $K > 0$ ,  $\eta = 1$ . Let  $\bar{\mu}_Y$  denote the  $\bar{F}$ -invariant probability measure on  $\bar{Y}$  from Proposition 2.5. We assume that  $\tau : \bar{Y} \rightarrow \mathbb{Z}^+$  is integrable. We suppose also that there is an  $F$ -invariant probability measure  $\mu_Y$  on  $Y$  such that  $\bar{\pi}_* \mu_Y = \bar{\mu}_Y$  where  $\bar{\pi} : Y \rightarrow \bar{Y}$  is the quotient map.

As in Subsection 2.2, starting from  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  and  $\tau : \bar{Y} \rightarrow \mathbb{Z}^+$ , we can form the *quotient tower*  $\bar{f} : \bar{\Delta} \rightarrow \bar{\Delta}$  with  $\bar{f}$ -invariant mixing probability measure  $\bar{\mu}_\Delta$ . Similarly, starting from  $F : Y \rightarrow Y$  and  $\tau : Y \rightarrow \mathbb{Z}^+$ , we form the tower  $f : \Delta \rightarrow \Delta$  such that  $F = f^\tau : Y \rightarrow Y$  with  $f$ -invariant mixing probability measure  $\mu_\Delta$ .

Define the semiconjugacy  $\pi : \Delta \rightarrow M$ ,  $\pi(y, \ell) = T^\ell y$ . Then  $\mu = \pi_* \mu_\Delta$  is a  $T$ -invariant mixing probability measure on  $M$ .

As in Subsection 2.2, we restrict to the cases  $\mu(\tau > n) = O(n^{-\beta})$ ,  $\beta > 1$ , and  $\mu(\tau > n) = O(e^{-An^\gamma})$ ,  $A > 0$ ,  $\gamma \in (0, 1]$ .

**Theorem 2.10** *Let  $\eta \in (0, 1]$ . Then there exists  $C > 0$ ,  $B > 0$  depending continuously on the constants in Theorem 2.7 (associated to the nonuniformly expanding map  $\bar{f} : \bar{\Delta} \rightarrow \bar{\Delta}$ ) as well as  $\eta$ ,  $\rho_0$  and  $K_0$ , such that  $|\int_M v w \circ T^n d\mu - \int_M v d\mu \int_M w d\mu| \leq C a_n \|v\|_\eta \|w\|_\eta$ , for all  $v, w \in C^\eta(M)$ ,  $n \geq 1$ , where  $a_n = n^{-(\beta-1)}$  or  $e^{-Bn^\gamma}$  respectively.*

**Remark 2.11** Note that there is no assumption about contraction rates along stable manifolds for  $T$ ; all that is required is exponential contraction/expansion for the

induced map  $F : Y \rightarrow Y$ . This is in contrast to [30] where exponential contraction is assumed for  $T$  (this restriction is also present in [6]) and [4] where polynomial contraction is assumed for  $T$ .

The method for removing such assumptions on contractivity of  $T$  are due to [16] (based on ideas in [10]) and were used previously in [23, Appendix B].

### 3 Proof for uniformly expanding maps

In this section, we prove Theorem 2.1 and Proposition 2.5.

For  $\psi : Y \rightarrow (0, \infty)$ , we define  $|\psi|_{\eta, \ell} = |\log \psi|_{\eta}$ . Note that

$$e^{-|\psi|_{\eta, \ell}} \int_Y \psi \, dm \leq \psi \leq e^{|\psi|_{\eta, \ell}} \int_Y \psi \, dm. \quad (3.1)$$

Also, for at most countably many observables  $\psi_k : Y \rightarrow (0, \infty)$ ,

$$\left| \sum_k \psi_k \right|_{\eta, \ell} \leq \sup_k |\psi_k|_{\eta, \ell}. \quad (3.2)$$

**Proposition 3.1** *Let  $\psi : Y \rightarrow (0, \infty)$ . Then  $|P_m \psi|_{\eta, \ell} \leq K + \lambda^{-\eta} |\psi|_{\eta, \ell}$ .*

**Proof** For  $a \in \alpha$  write  $\psi_a = 1_a \psi$ . Then  $P_m \psi = \sum_a P_m \psi_a$ . For  $y \in Y$ , we have  $(P_m \psi_a)(y) = \zeta(y_a) \psi(y_a)$  where  $y_a$  is the unique preimage of  $y$  under  $F$  lying in  $a$ .

Let  $x, y \in Y$  with preimages  $x_a, y_a \in a$ . Then

$$\begin{aligned} |\log(P_m \psi_a)(x) - \log(P_m \psi_a)(y)| &\leq |\log \zeta(x_a) - \log \zeta(y_a)| + |\log \psi(x_a) - \log \psi(y_a)| \\ &\leq K d(Fx_a, Fy_a)^\eta + |\psi|_{\eta, \ell} d(x_a, y_a)^\eta \leq (K + \lambda^{-\eta} |\psi|_{\eta, \ell}) d(x, y)^\eta, \end{aligned}$$

and so  $|P_m \psi_a|_{\eta, \ell} \leq K + \lambda^{-\eta} |\psi|_{\eta, \ell}$ . The result follows from (3.2). ■

**Proposition 3.2** *Let  $\psi : Y \rightarrow (0, \infty)$ . For each  $t \in [0, e^{-|\psi|_{\eta, \ell}}]$*

$$\left| \psi - t \int_Y \psi \, dm \right|_{\eta, \ell} \leq \frac{|\psi|_{\eta, \ell}}{1 - t e^{|\psi|_{\eta, \ell}}}.$$

**Proof** Let  $\kappa(y) = \log \psi(y)$ . Note that

$$\frac{d}{d\kappa} \log \left( e^\kappa - t \int_Y \psi \, dm \right) = \frac{e^\kappa}{e^\kappa - t \int_Y \psi \, dm} = \frac{1}{1 - t e^{-\kappa} \int_Y \psi \, dm}.$$

By (3.1),

$$\frac{1}{1 - t e^{-\kappa(y)} \int_Y \psi \, dm} = \frac{1}{1 - t \psi(y)^{-1} \int_Y \psi \, dm} \leq \frac{1}{1 - t e^{|\psi|_{\eta, \ell}}},$$



for all  $y \in Y$ . Hence, by the mean value theorem, for  $x, y \in Y$ ,

$$\left| \log \left( e^{\kappa(x)} - t \int_Y \psi \, dm \right) - \log \left( e^{\kappa(y)} - t \int_Y \psi \, dm \right) \right| \leq \frac{|\kappa(x) - \kappa(y)|}{1 - te^{|\psi|_{\eta, \ell}}} \leq \frac{|\psi|_{\eta, \ell} d(x, y)^\eta}{1 - te^{|\psi|_{\eta, \ell}}}.$$

This completes the proof.  $\blacksquare$

Fix constants  $R > 0$  and  $\xi \in (0, e^{-R})$ , such that  $R(1 - \xi e^R) \geq K + \lambda^{-\eta} R$ . (For example, choose  $R$  and  $\xi$  as in Remark 2.2.)

**Proposition 3.3** *Let  $\psi : Y \rightarrow (0, \infty)$  with  $|\psi|_{\eta, \ell} \leq R$ . Then  $|P_m \psi|_{\eta, \ell} \leq R$ .*

**Proof** By Proposition 3.1,  $|P_m \psi|_{\eta, \ell} \leq K + \lambda^{-\eta} R \leq R$ .  $\blacksquare$

**Lemma 3.4** *Let  $\psi_1, \psi_2 : Y \rightarrow (0, \infty)$  with  $|\psi_1|_{\eta, \ell} \leq R$ ,  $|\psi_2|_{\eta, \ell} \leq R$ , and  $\int_Y \psi_1 \, dm = \int_Y \psi_2 \, dm$ . Let  $\psi'_j = P_m \psi_j - \xi \int_Y \psi_j \, dm$  for  $j = 1, 2$ . Then*

- (a)  $|\psi'_j|_{\eta, \ell} \leq R$  for  $j = 1, 2$ ,
- (b)  $P_m \psi_1 - P_m \psi_2 = \psi'_1 - \psi'_2$ ,
- (c)  $\int_Y \psi'_1 \, dm = \int_Y \psi'_2 \, dm = (1 - \xi) \int_Y \psi_1 \, dm$ .

**Proof** By Propositions 3.1 and 3.2,

$$|\psi'_j|_{\eta, \ell} = \left| P_m \psi_j - \xi \int_Y \psi_j \, dm \right|_{\eta, \ell} \leq \frac{|P_m \psi_j|_{\eta, \ell}}{1 - \xi e^{|P_m \psi_j|_{\eta, \ell}}} \leq \frac{K + \lambda^{-\eta} R}{1 - \xi e^R} \leq R,$$

proving part (a). Parts (b) and (c) are immediate.  $\blacksquare$

Now we are ready to prove Theorem 2.1 taking  $C = 4e^R(1 + R)$  and  $\gamma = 1 - \xi$ .

**Proof of Theorem 2.1** Assume first that  $|\phi|_\eta \leq R$ . Later we remove this restriction.

Since  $\int_Y \phi \, dm = 0$ , there exists  $x, y \in Y$  such that  $\phi(x) \leq 0 \leq \phi(y)$ . Hence it follows from the assumption  $|\phi|_\eta \leq R$  that  $|\phi|_\infty \leq R$ .

Write  $\phi = \psi_0^+ - \psi_0^-$ , where  $\psi_0^+ = 1 + \max\{0, \phi\}$  and  $\psi_0^- = 1 - \min\{0, \phi\}$ . Then  $\psi_0^\pm : Y \rightarrow [1, \infty)$  and  $\int_Y \psi_0^+ \, d\mu = \int_Y \psi_0^- \, d\mu \leq 1 + |\phi|_\infty \leq 1 + R$ . For  $x, y \in Y$ ,

$$|\log \psi_0^\pm(x) - \log \psi_0^\pm(y)| \leq |\psi_0^\pm(x) - \psi_0^\pm(y)| \leq |\phi(x) - \phi(y)|,$$

so  $|\psi_0^\pm|_{\eta, \ell} \leq |\phi|_\eta \leq R$ .

Define

$$\psi_{n+1}^\pm = P_m \psi_n^\pm - \xi \int_Y \psi_n^\pm \, dm, \quad n \geq 0.$$

By Lemma 3.4(a),  $|\psi_n^\pm|_{\eta,\ell} \leq R$  for all  $n \geq 0$ . By Lemma 3.4(b,c),

$$P_m^n \phi = P_m^n \psi_0^+ - P_m^n \psi_0^- = \psi_n^+ - \psi_n^-, \quad (3.3)$$

and  $\int_Y \psi_n^\pm dm = \gamma^n \int_Y \psi_0^\pm dm \leq (1+R)\gamma^n$ . By (3.1),

$$\psi_n^\pm \leq e^R \int_Y \psi_n^\pm dm \leq e^R(1+R)\gamma^n. \quad (3.4)$$

Next, we recall the inequality

$$|a - b| \leq \max\{a, b\} |\log a - \log b|, \text{ for all } a, b > 0. \quad (3.5)$$

By (3.5) and the definition of  $|\psi|_{\eta,\ell}$ , for  $x, y \in Y$ ,

$$\begin{aligned} |\psi_n^\pm(x) - \psi_n^\pm(y)| &\leq \max(\psi_n^\pm(x), \psi_n^\pm(y)) |\log \psi_n^\pm(x) - \log \psi_n^\pm(y)| \\ &\leq e^R(1+R)\gamma^n |\psi_n^\pm|_{\eta,\ell} d(x, y)^\eta \leq e^R R(1+R)\gamma^n d(x, y)^\eta. \end{aligned}$$

Hence,  $|\psi_n^\pm|_\eta \leq e^R R(1+R)\gamma^n$ . By (3.3),

$$|P_m^n \phi|_\eta \leq 2e^R R(1+R)\gamma^n. \quad (3.6)$$

Finally, we remove the restriction  $|\phi|_\eta \leq R$ . Note that  $u = R|\phi|_\eta^{-1}\phi$  satisfies  $|u|_\eta \leq R$ , and therefore it follows from (3.6) that

$$|P_m^n \phi|_\eta = R^{-1} |\phi|_\eta |P_m^n u|_\eta \leq 2e^R(1+R)\gamma^n |\phi|_\eta.$$

Also,  $\int_Y P_m^n \phi dm = 0$ , so  $|P_m^n \phi|_\infty \leq |P_m^n \phi|_\eta$ . Hence

$$\|P_m^n \phi\|_\eta \leq 2|P_m^n \phi|_\eta \leq 4e^R(1+R)\gamma^n |\phi|_\eta,$$

as required. ■

**Proof of Proposition 2.5** We construct an invariant probability measure  $\mu \in \mathcal{M}$  and show that  $|\frac{d\mu}{dm}|_{\eta,\ell} \leq R$ .

By Proposition 3.3,  $|P_m^n 1|_{\eta,\ell} \leq R$  for all  $n \geq 0$ . In particular, it follows from (3.1) that  $|P_m 1|_\infty \leq e^R$ . By (3.5),

$$|P_m 1|_\eta \leq |P_m 1|_\infty |P_m 1|_{\eta,\ell} \leq e^R R.$$

Also,  $\int_Y (P_m 1 - 1) dm = 0$ , so by Theorem 2.1,  $\|P_m^n (P_m 1 - 1)\|_\eta \leq C e^R R \gamma^n$ . Hence we can define

$$\rho = \lim_{n \rightarrow \infty} P_m^n 1 = 1 + \sum_{n=0}^{\infty} P_m^n (P_m 1 - 1) \in C^\eta.$$

It is immediate that  $\int_Y \rho dm = 1$  and  $P_m \rho = \rho$ , so  $\rho$  is an invariant density. Moreover, for  $x, y \in Y$ ,

$$|\log \rho(x) - \log \rho(y)| = \lim_{n \rightarrow \infty} |\log(P_m^n 1)(x) - \log(P_m^n 1)(y)| \leq R d(x, y)^\eta,$$

so that  $|\rho|_{\eta,\ell} \leq R$ . ■

**Remark 3.5** In this paper, we have restricted attention to expanding maps  $F : Y \rightarrow Y$  satisfying the full branch condition  $Fa = Y$  for all  $a \in \alpha$ . This is a reasonable restriction for situations where the expanding maps are obtained by inducing nonuniformly expanding maps as in [18]. More generally, the restriction is justified by the family of examples  $F_\delta : [0, 1] \rightarrow [0, 1]$  depicted in Figure 1 below. Note that each map preserves Lebesgue measure and is mixing. Moreover, we can take  $\lambda = 2$  and  $K = 0$  for all  $\delta$ . Nevertheless, correlations decay arbitrarily slowly as  $\delta \rightarrow 0$ . (Explicit constants depending on  $\delta$  can be computed from [32].)

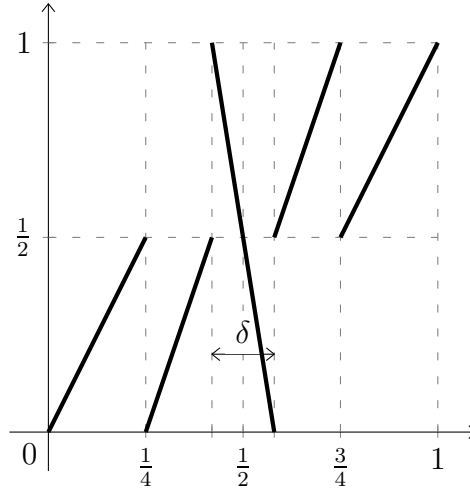


Figure 1: A family of uniformly expanding maps  $F_\delta : [0, 1] \rightarrow [0, 1]$  with  $\lambda = 2$  and  $K = 0$  but with arbitrarily slow decay of correlations.

## 4 Proof for nonuniformly expanding maps

In this section, we prove Theorems 2.7 and 2.8.

### 4.1 Outline of the proof

Let  $\Delta_\ell = \{(y, k) \in \Delta : k = \ell\}$  denote the  $\ell$ -th level of the tower. Our strategy is to construct a countable probability space  $(W, \mathbb{P})$  and a random variable  $h : W \rightarrow \mathbb{N}$  such that every sufficiently regular observable  $\psi : \Delta \rightarrow [0, \infty)$  with  $\int_\Delta \psi dm_\Delta = 1$  can be decomposed into a series  $\psi = \sum_{w \in W} \psi_w$  where  $\psi_w : \Delta \rightarrow [0, \infty)$  are such that  $\int_\Delta \psi_w dm_\Delta = \mathbb{P}(w)$  and  $L^{h(w)}\psi_w = \mathbb{P}(w)\bar{\tau}1_{\Delta_0}$ .

Now let  $\phi : \Delta \rightarrow \mathbb{R}$  and suppose that  $L^N\phi = C(\psi - \psi')$  where  $\psi$  and  $\psi'$  can be decomposed as above and  $C > 0$ ,  $N \in \mathbb{N}$  are constants. We have  $L^{h(w)}\psi_w = L^{h(w)}\psi'_w$ ,

and so  $L^n(\psi_w - \psi'_w) = 0$  whenever  $n \geq h(w)$ . Therefore

$$\begin{aligned} \int_{\Delta} |L^{N+n}\phi| dm_{\Delta} &\leq C \sum_{w \in W: h(w) > n} \int_{\Delta} (L^n \psi_w + L^n \psi'_w) dm_{\Delta} \\ &= C \sum_{w \in W: h(w) > n} \int_{\Delta} (\psi_w + \psi'_w) dm_{\Delta} = 2C\mathbb{P}(h > n). \end{aligned}$$

In this way, decay rates for  $L^n\phi$  reduce to tail estimates for  $h$ .

## 4.2 Recurrence to $\Delta_0$

Given  $\psi : \Delta \rightarrow [0, \infty)$ , define

$$|\psi|_{\eta, \ell} = \sup_{n \geq 0} \sup_{(y, n) \neq (y', n) \in \Delta_n} \frac{|\log \psi(y, n) - \log \psi(y', n)|}{d(y, y')^{\eta}},$$

where  $\log 0 = -\infty$  and  $\log 0 - \log 0 = 0$ .

As in Section 3, we fix constants  $R > 0$  and  $\xi \in (0, e^{-R})$ , such that  $R(1 - \xi e^R) \geq K + \lambda^{-\eta}R$ . (For example, choose  $R$  and  $\xi$  as in Remark 2.2.) Using notation from Section 3,  $(L\phi)(y, \ell) = \begin{cases} \phi(y, \ell - 1) & \ell \geq 1 \\ \sum_{a \in \alpha} \zeta(y_a) \phi(y_a, \tau(a) - 1) & \ell = 0 \end{cases}$ .

**Proposition 4.1** *Let  $\psi : \Delta \rightarrow [0, \infty)$  with  $|\psi|_{\eta, \ell} \leq R$ . Then*

$$(a) \quad e^{-R\bar{\tau}} \int_{\Delta_0} \psi dm_{\Delta} \leq \psi 1_{\Delta_0} \leq e^{R\bar{\tau}} \int_{\Delta_0} \psi dm_{\Delta}.$$

$$(b) \quad |L\psi|_{\eta, \ell} \leq R.$$

$$(c) \quad \text{If } t \in [0, \xi], \text{ then } \psi' = L\psi - t\bar{\tau} \int_{\Delta_0} L\psi dm_{\Delta} 1_{\Delta_0} \text{ is nonnegative and } |\psi'|_{\eta, \ell} \leq R.$$

**Proof** (a) This is the counterpart of (3.1).

(b) Let  $(y, \ell), (y', \ell) \in \Delta_{\ell}$ . If  $\ell \geq 1$ , then it is immediate that  $|\log(L\psi)(y, \ell) - \log(L\psi)(y', \ell)| \leq Rd(y, y')^{\eta}$ . The same calculation as in Proposition 3.1 shows that

$$|\log(L\psi)(y, 0) - \log(L\psi)(y', 0)| \leq (K + \lambda^{-\eta}R)d(y, y')^{\eta} \leq Rd(y, y')^{\eta}.$$

(c) It follows from (b) that  $|L\psi|_{\eta, \ell} \leq R$ . Hence, by (a),  $\psi' \geq 0$ . As in part (b), it is immediate that  $|\log \psi'(y, \ell) - \log \psi'(y', \ell)| \leq Rd(y, y')^{\eta}$  for  $\ell \geq 1$ . Also,  $\psi'(y, 0) = \chi(y) - t \int_Y \chi dm$  where  $\chi : Y \rightarrow [0, \infty)$  is given by  $\chi(y) = (L\psi)(y, 0)$ , so it follows from Proposition 3.2 that  $|\log \psi'(y, 0) - \log \psi'(y', 0)| \leq (K + \lambda^{-\eta}R)(1 - te^R)^{-1}d(y, y')^{\eta} \leq Rd(y, y')^{\eta}$ .  $\blacksquare$

Define  $N = N_1 + N_2$  where

$$N_1 = \max\{I_k^2\}, \quad N_2 = \min\{n \geq 1 : m_\Delta(\cup_{\ell \geq n} \Delta_\ell) \leq \frac{1}{2}e^{-R\bar{\tau}^{-1}}\}.$$

Let  $\mathcal{A}$  be the set of observables  $\psi : \Delta \rightarrow [0, \infty)$  such that  $|\psi|_\infty \leq e^{R\bar{\tau}} \int_\Delta \psi dm_\Delta$  and  $|\psi|_{\eta, \ell} \leq R$ . Define  $\mathcal{B} = L^N \mathcal{A}$ .

**Corollary 4.2** (a) If  $\psi : \Delta \rightarrow [0, \infty)$  is supported on  $\Delta_0$ , and  $|\psi|_{\eta, \ell} \leq R$ , then  $\psi \in \mathcal{A}$ .

(b) If  $\psi, \psi' \in \mathcal{A}$  (or  $\mathcal{B}$ ) and  $t \geq 0$ , then  $L\psi$ ,  $\psi + \psi'$  and  $t\psi$  belong in  $\mathcal{A}$  (or  $\mathcal{B}$ ). In particular,  $\mathcal{B} \subset \mathcal{A}$ .

**Proof** Part (a) follows from Proposition 4.1(a). Next, let  $\psi \in \mathcal{A}$ . We show that  $L\psi \in \mathcal{A}$ ; the remaining statements in part (b) are immediate. By Proposition 4.1(b),  $|L\psi|_{\eta, \ell} \leq R$ . Also, using the definition of  $\mathcal{A}$  and Proposition 4.1(a),

$$\begin{aligned} |1_{\Delta \setminus \Delta_0} L\psi|_\infty &\leq |\psi|_\infty \leq e^{R\bar{\tau}} \int_\Delta \psi dm_\Delta = e^{R\bar{\tau}} \int_\Delta L\psi dm_\Delta, \quad \text{and} \\ |1_{\Delta_0} L\psi|_\infty &\leq e^{R\bar{\tau}} \int_{\Delta_0} L\psi dm_\Delta \leq e^{R\bar{\tau}} \int_\Delta L\psi dm_\Delta. \end{aligned}$$

Hence  $|L\psi|_\infty \leq e^{R\bar{\tau}} \int_\Delta L\psi dm_\Delta$ , so  $L\psi \in \mathcal{A}$ . ■

**Proposition 4.3** If  $\psi \in \mathcal{A}$ , then  $\max_{0 \leq j \leq N_2} \int_{\Delta_0} L^j \psi dm_\Delta \geq \frac{1}{2}e^{-R\bar{\tau}^{-1}} \int_\Delta \psi dm_\Delta$ .

**Proof** It follows from the definition of  $N_2$  and  $\mathcal{A}$  that  $m_\Delta(\cup_{\ell=N_2+1}^\infty \Delta_\ell) |\psi|_\infty \leq \frac{1}{2} \int_\Delta \psi dm_\Delta$ . Hence

$$\begin{aligned} \int_\Delta \psi dm_\Delta &= \int_{\cup_{\ell=0}^{N_2} \Delta_\ell} L^{N_2} \psi dm_\Delta + \int_{\cup_{\ell=N_2+1}^\infty \Delta_\ell} L^{N_2} \psi dm_\Delta \\ &\leq \int_{\cup_{\ell=0}^{N_2} \Delta_\ell} L^{N_2} \psi dm_\Delta + \frac{1}{2} \int_\Delta \psi dm_\Delta, \end{aligned}$$

so  $\int_{\cup_{\ell=0}^{N_2} \Delta_\ell} L^{N_2} \psi dm_\Delta \geq \frac{1}{2} \int_\Delta \psi dm_\Delta$ .

Next, if  $\ell \leq N_2$  then  $(L^{N_2} \psi)(y, \ell) = (L^{N_2-\ell} \psi)(y, 0)$ , and so  $\int_{\Delta_\ell} L^{N_2} \psi dm_\Delta \leq m_\Delta(\Delta_\ell) |L^{N_2-\ell} \psi 1_{\Delta_0}|_\infty \leq m_\Delta(\Delta_\ell) \max_{0 \leq j \leq N_2} |L^j \psi 1_{\Delta_0}|_\infty$ . Hence, by Proposition 4.1(a,b),

$$\int_{\cup_{\ell=0}^{N_2} \Delta_\ell} L^{N_2} \psi dm_\Delta \leq \max_{0 \leq j \leq N_2} |L^j \psi 1_{\Delta_0}|_\infty \leq e^{R\bar{\tau}} \max_{0 \leq j \leq N_2} \int_{\Delta_0} L^j \psi dm_\Delta.$$

The result follows. ■

**Proposition 4.4** *If  $|\psi|_{\eta,\ell} \leq R$ , then  $\int_{\Delta_0} L^n \psi dm_\Delta \geq (e^{-R}\delta)^n \int_{\Delta_0} \psi dm_\Delta$  for all  $n \geq N_1$ .*

**Proof** By Proposition 4.1(a),  $\inf_{\Delta_0} \psi \geq e^{-R\bar{\tau}} \int_{\Delta_0} \psi dm_\Delta$ . By our assumptions,  $m_\Delta(\{x \in \Delta_0 : f^{I_k} x \in \Delta_0\}) \geq \delta/\bar{\tau}$  for every  $I_k$ . Hence

$$\begin{aligned} \int_{\Delta_0} L^{I_k} \psi dm_\Delta &= \int_{\Delta} \psi 1_{\Delta_0} \circ f^{I_k} dm_\Delta \geq \int_{\Delta_0} \psi 1_{\Delta_0} \circ f^{I_k} dm_\Delta \\ &\geq \inf_{\Delta_0} \psi m_\Delta(\{x \in \Delta_0 : f^{I_k} x \in \Delta_0\}) \geq e^{-R}\delta \int_{\Delta_0} \psi dm_\Delta. \end{aligned}$$

By [26], every  $n \geq N_1$  can be written as  $n = \sum_k n_k I_k$ , where  $n_k$  are nonnegative integers. By Proposition 4.1(b), it follows inductively that

$$\int_{\Delta_0} L^n \psi dm_\Delta \geq (e^{-R}\delta)^{\sum_k n_k} \int_{\Delta_0} \psi dm_\Delta \geq (e^{-R}\delta)^n \int_{\Delta_0} \psi dm_\Delta,$$

as required. ■

**Lemma 4.5** *If  $\psi \in \mathcal{B}$ , then  $\int_{\Delta_0} \psi dm_\Delta \geq \epsilon \int_{\Delta} \psi dm_\Delta$ , where  $\epsilon = \frac{1}{2}e^{-R\bar{\tau}-1}(e^{-R}\delta)^{N_1}$ .*

**Proof** By definition of  $\mathcal{B}$ , there exists  $\psi' \in \mathcal{A}$  such that  $L^{N_1+N_2}\psi' = \psi$ . By Proposition 4.3, there exists  $j \leq N_2$  such that  $\int_{\Delta_0} L^j \psi' dm_\Delta \geq \frac{1}{2}e^{-R\bar{\tau}-1} \int_{\Delta} \psi' dm_\Delta$ . By Proposition 4.4 (taking  $n = N_1 + N_2 - j \geq N_1$ ),

$$\begin{aligned} \int_{\Delta_0} \psi dm_\Delta &= \int_{\Delta_0} L^{N_1+N_2} \psi' dm_\Delta \geq (e^{-R}\delta)^{N_1+N_2-j} \int_{\Delta_0} L^j \psi' dm_\Delta \\ &\geq \frac{1}{2}e^{-R\bar{\tau}-1}(e^{-R}\delta)^{N_1+N_2} \int_{\Delta} \psi' dm_\Delta = \epsilon \int_{\Delta} \psi dm_\Delta, \end{aligned}$$

as required. ■

### 4.3 Decomposition in $\mathcal{B}$

Next, we introduce constants  $p_n, t_n \in [0, 1]$ ,

$$\begin{aligned} t_1 &= 1 - \epsilon, \quad t_n = \min\{t_1, e^{R\bar{\tau}} m_\Delta(\cup_{\ell=n}^\infty \Delta_\ell)\}, \quad n \geq 2, \\ p_{-1} &= \xi\epsilon, \quad p_0 = (1 - \xi)\epsilon, \quad p_n = t_n - t_{n+1}, \quad n \geq 1. \end{aligned}$$

The monotonicity of the sequence  $t_n$  ensures that  $p_n \geq 0$  for all  $n$ . Note that  $\sum_{n=-1}^\infty p_n = 1$ .

Let  $E_0 = \Delta_0$  and  $E_k = \{(y, \ell) \in \Delta : \ell = \tau(y) - k, \ell \geq 1\}$  for  $k \geq 1$ . Then  $\{E_0, E_1, \dots\}$  defines a partition of  $\Delta$  and  $m_\Delta(E_k) = m_\Delta(\Delta_k)$  for all  $k$ .

**Proposition 4.6** *If  $\psi \in \mathcal{B}$  with  $\int_{\Delta} \psi dm_{\Delta} = 1$ , then  $\int_{\bigcup_{\ell=n}^{\infty} E_{\ell}} \psi dm_{\Delta} \leq t_n$ , for  $n \geq 1$ .*

**Proof** By Lemma 4.5,  $\int_{\bigcup_{\ell=n}^{\infty} E_{\ell}} \psi dm_{\Delta} \leq \int_{\bigcup_{\ell=1}^{\infty} E_{\ell}} \psi dm_{\Delta} \leq 1 - \epsilon = t_1$  for all  $n \geq 1$ .

By definition of  $\mathcal{B}$ , for  $n \geq 2$  we have in addition that  $\int_{\bigcup_{\ell=n}^{\infty} E_{\ell}} \psi dm_{\Delta} \leq m_{\Delta}(\bigcup_{\ell=n}^{\infty} \Delta_{\ell})|\psi|_{\infty} \leq e^R \bar{\tau} m_{\Delta}(\bigcup_{\ell=n}^{\infty} \Delta_{\ell})$ . The result follows by definition of  $t_n$ . ■

**Proposition 4.7** *Let  $p_j, q_j \in [0, \infty)$  be sequences such that  $\sum_{j=0}^{\infty} p_j = \sum_{j=0}^{\infty} q_j < \infty$  and  $\sum_{j=0}^k q_j \geq \sum_{j=0}^k p_j$  for all  $k \geq 0$ . Then there exist  $s_{k,j} \in [0, 1]$ ,  $0 \leq j \leq k$ , such that  $\sum_{j=0}^k s_{k,j} q_j = p_k$  for all  $k \geq 0$  and  $\sum_{k=j}^{\infty} s_{k,j} = 1$  for all  $j \geq 0$ .*

**Proof** We assume that  $q_j > 0$  for all  $j$ ; otherwise set  $s_{k,j} = \delta_{k,j}$  for  $k \leq j$  whenever  $q_j = 0$ .

For  $k = 0$ , choose  $s_{0,0} = p_0/q_0$ . Next let  $k \geq 1$ , and suppose inductively that  $s_{k',j}$  have been constructed for  $0 \leq j \leq k' \leq k-1$ , such that  $\sum_{j=0}^{k'} s_{k',j} q_j = p_{k'}$  for  $k' \leq k-1$  and  $\sum_{k'=j}^{k-1} s_{k',j} \leq 1$  for  $j \leq k-1$ .

Define  $s_{k,0}, s_{k,1}, \dots, s_{k,k} \in [0, 1]$  (in this order) by

$$s_{k,j} = \min \left\{ 1 - \sum_{k'=j}^{k-1} s_{k',j}, \frac{p_k - \sum_{j'=0}^{j-1} s_{k,j'} q_{j'}}{q_j} \right\}, \quad j = 0, 1, \dots, k.$$

By construction,  $\sum_{j=0}^k s_{k,j} q_j \leq p_k$ . If  $\sum_{j=0}^k s_{k,j} q_j < p_k$ , then necessarily  $\sum_{k'=j}^k s_{k',j} = 1$  for all  $j \leq k$ , and so

$$\sum_{k'=0}^k q_{k'} = \sum_{k'=0}^k \sum_{j=0}^{k'} s_{k',j} q_j = \sum_{j=0}^k s_{k,j} q_j + \sum_{k'=0}^{k-1} p_{k'} < \sum_{k'=0}^k p_{k'},$$

which is a contradiction. Hence  $\sum_{j=0}^k s_{k,j} q_j = p_k$ .

By the above construction,  $\sum_{j=0}^k s_{k,j} q_j = p_k$  for  $k \geq 0$  and  $\sum_{k=j}^{\infty} s_{k,j} \leq 1$  for  $j \geq 0$ . Since also  $\sum_{j=0}^{\infty} p_j = \sum_{j=0}^{\infty} q_j < \infty$ , we conclude that  $\sum_{k=j}^{\infty} s_{k,j} = 1$ . ■

**Lemma 4.8** *Let  $\psi : \Delta \rightarrow [0, \infty)$  be such that  $L^n \psi \in \mathcal{B}$  for some  $n \geq 0$ . Then  $\psi = \sum_{k=-1}^{\infty} \psi_k$ , where  $\psi_k : \Delta \rightarrow [0, \infty)$  are such that*

$$(i) \quad L^n \psi_{-1} = p_{-1} \bar{\tau} \int_{\Delta} \psi dm_{\Delta} 1_{\Delta_0}, \quad (ii) \quad L^{k+n} \psi_k \in \mathcal{A} \text{ for all } k \geq 0,$$

$$(iii) \quad \int_{\Delta} \psi_k dm_{\Delta} = p_k \int_{\Delta} \psi dm_{\Delta} \text{ for all } k \geq -1.$$

**Proof** First we consider the case  $n = 0$ . Suppose without loss that  $\int_{\Delta} \psi dm_{\Delta} = 1$ . Define  $\psi_{-1} = p_{-1} \bar{\tau} 1_{\Delta_0}$  in accordance with properties (i) and (iii).

By Lemma 4.5,  $\int_{\Delta_0} \psi dm_\Delta \geq \epsilon$ . Hence  $t = p_{-1} / \int_{\Delta_0} \psi dm_\Delta = \xi\epsilon / \int_{\Delta_0} \psi dm_\Delta \in [0, \xi]$ . Since  $\psi \in \mathcal{B} \subset L\mathcal{A}$ , it follows from Proposition 4.1(c) that

$$\psi' = \psi - t\bar{\tau} \int_{\Delta_0} \psi dm_\Delta 1_{\Delta_0} = \psi - p_{-1}\bar{\tau}1_{\Delta_0} = \psi - \psi_{-1}$$

is nonnegative and  $|\psi'|_{\eta,\ell} \leq R$ . Setting  $g_0 = \psi'1_{\Delta_0}$ , we obtain that  $\psi1_{\Delta_0} = \psi_{-1} + g_0$  where  $g_0$  is nonnegative and  $|g_0|_{\eta,\ell} \leq R$ . By Corollary 4.2(a),  $g_0 \in \mathcal{A}$ .

Define  $g_k = \psi1_{E_k}$  for  $k \geq 1$ . Note that  $L^k g_k$  is supported on  $\Delta_0$  and  $|L^k g_k|_{\eta,\ell} \leq R$ . By Corollary 4.2(a),  $L^k g_k \in \mathcal{A}$ .

Now  $\psi = \psi1_{\Delta_0} + \sum_{k=1}^{\infty} g_k = \psi_{-1} + \sum_{k=0}^{\infty} g_k$ . By Proposition 4.6,

$$p_{-1} + \sum_{j=0}^k \int_{\Delta} g_j dm_\Delta = 1 - \sum_{j=k+1}^{\infty} \int_{\Delta} g_j dm_\Delta \geq 1 - t_{k+1} = \sum_{j=-1}^k p_j.$$

Setting  $q_k = \int_{\Delta} g_k dm_\Delta$ , we have that  $\sum_{j=0}^k q_j \geq \sum_{j=0}^k p_j$  for all  $k \geq 0$ . Choose  $s_{k,j} \in [0, 1]$  as in Proposition 4.7, and define  $\psi_k : \Delta \rightarrow [0, \infty)$ ,  $k \geq 0$ , by

$$\psi_k = \sum_{j=0}^k s_{k,j} g_j.$$

By construction, condition (iii) holds for all  $k$ . Condition (ii) is satisfied by Corollary 4.2(b). Finally, by Proposition 4.7,

$$\sum_{k=0}^{\infty} \psi_k = \sum_{k=0}^{\infty} \sum_{j=0}^k s_{k,j} g_j = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} s_{k,j} g_j = \sum_{j=0}^{\infty} g_j = \psi - \psi_{-1},$$

completing the proof for  $n = 0$ .

Now suppose  $L^n \psi \in \mathcal{B}$  for some  $n \geq 1$ . Setting  $\psi' = L^n \psi$  and applying the result for  $n = 0$ , we can write  $\psi' = \sum_{k=-1}^{\infty} \psi'_k$  where  $\psi'_k$  satisfy properties (i)–(iii). Define  $\psi_k = (\frac{\psi'_k}{\psi'} \circ f^n) \psi$  with the convention that  $0/0 = 0$ . Then  $L^n \psi_k = \frac{\psi'_k}{\psi'} L^n \psi = \psi'_k$ , so properties (i)–(iii) are passed down from  $\psi'_k$  to  $\psi_k$ . Also  $\sum_{k=-1}^{\infty} \psi_k = (\frac{\psi'}{\psi'} \circ f^n) \psi = \psi$ . ■

Let  $W$  be the countable set of all finite words in the alphabet  $\mathbb{N} = \{0, 1, 2, \dots\}$  including the zero length word, and let  $W_k$  be the subset consisting of words of length  $k$ . Let  $\mathbb{P}$  be the probability measure on  $W$  given for  $w = w_1 \cdots w_k \in W$  by  $\mathbb{P}(w) = p_{-1}p_{w_1} \cdots p_{w_k}$ . Define  $h : W \rightarrow \mathbb{N}$  by  $h(w) = \Sigma w + N|w|$ , where  $\Sigma w = w_1 + \cdots + w_k$  and  $|w| = k$  for  $w = w_1 \cdots w_k$ .

**Proposition 4.9** *Let  $\psi \in \mathcal{B}$  with  $\int_{\Delta} \psi dm_\Delta = 1$ . Then  $\psi = \sum_{w \in W} \psi_w$ , where  $\psi_w : \Delta \rightarrow [0, \infty)$  are such that  $\int_{\Delta} \psi_w dm_\Delta = \mathbb{P}(w)$  and  $L^{h(w)} \psi_w = \mathbb{P}(w)\bar{\tau}1_{\Delta_0}$ .*

**Proof** Write  $\psi = \sum_{k=-1}^{\infty} \psi_k$  as in Lemma 4.8 (with  $n = 0$ ). By properties (iii) and (i),  $\int_{\Delta} \psi_k dm_\Delta = p_k$  for all  $k \geq -1$ , and  $\psi_{-1} = p_{-1}\bar{\tau}1_{\Delta_0}$ .



Also  $L^{k+N}\psi_k \in \mathcal{B}$  by property (ii), allowing us to apply Lemma 4.8 to each  $\psi_k$  (with  $n = k + N$ ), yielding

$$\psi = \psi_{-1} + \sum_{k=0}^{\infty} \psi_k = \psi_{-1} + \sum_{k=0}^{\infty} \left( \psi_{-1,k} + \sum_{j=0}^{\infty} \psi_{j,k} \right),$$

where

$$\begin{aligned} \int_{\Delta} \psi_{j,k} dm_{\Delta} &= p_j \int_{\Delta} \psi_k dm_{\Delta} = p_j p_k, \\ L^{k+N} \psi_{-1,k} &= p_{-1} \bar{\tau} \int_{\Delta} \psi_k dm_{\Delta} 1_{\Delta_0} = p_{-1} p_k \bar{\tau} 1_{\Delta_0}. \end{aligned}$$

At the next step,

$$\begin{aligned} \psi &= \psi_{-1} + \sum_{k=0}^{\infty} \psi_{-1,k} + \sum_{j,k=0}^{\infty} \left( \psi_{-1,j,k} + \sum_{i=0}^{\infty} \psi_{i,j,k} \right), \\ &= \psi_{-1} + \sum_{w \in W_1} \psi_{-1,w} + \sum_{w \in W_2} \psi_{-1,w} + \sum_{i,j,k=0}^{\infty} \psi_{i,j,k}, \end{aligned}$$

where

$$\int_{\Delta} \psi_{i,j,k} = p_i p_j p_k, \quad L^{j+k+2N} \psi_{-1,j,k} = p_{-1} p_j p_k \bar{\tau} 1_{\Delta_0}.$$

In particular, for the terms  $\psi_{-1,w}$  with  $w \in W_0 \cup W_1 \cup W_2$ , we have the required properties  $\int_{\Delta} \psi_{-1w} dm_{\Delta} = \mathbb{P}(w)$  and  $L^{h(w)} \psi_{-1,w} = \mathbb{P}(w) \bar{\tau} 1_{\Delta_0}$ .

In this way we obtain  $\psi = \sum_{w \in W} \psi_{-1w}$  where  $\int_{\Delta} \psi_{-1w} dm_{\Delta} = \mathbb{P}(w)$  and  $L^{h(w)} \psi_{-1,w} = \mathbb{P}(w) \bar{\tau} 1_{\Delta_0}$ .  $\blacksquare$

#### 4.4 Proof of Theorems 2.7 and 2.8

Let  $\phi : \Delta \rightarrow \mathbb{R}$  be an observable as in Theorem 2.7, i.e.  $\|\phi\|_{\eta} < \infty$  and  $\int_{\Delta} \phi dm_{\Delta} = 0$ . Define  $\tilde{\psi}, \tilde{\psi}' : \Delta \rightarrow [0, \infty)$  by

$$\tilde{\psi} = 1 + \frac{\phi}{\|\phi\|_{\eta}(1 + R^{-1})}, \quad \tilde{\psi}' \equiv 1.$$

Then  $L^n \phi = \|\phi\|_{\eta}(1 + R^{-1})(\psi - \psi')$ , where  $\psi = L^N \tilde{\psi}$ ,  $\psi' = L^N \tilde{\psi}'$ .

Now  $\int_{\Delta} \psi dm_{\Delta} = \int_{\Delta} \psi' dm_{\Delta} = 1$ . Next,

$$|\tilde{\psi}|_{\infty} \leq 1 + \frac{1}{1 + R^{-1}} \leq 1 + R \leq e^R \leq \bar{\tau} e^R.$$

Also,  $|\tilde{\psi}| \geq 1 - (1 + R^{-1})^{-1} = (1 + R)^{-1}$  and  $|\tilde{\psi}|_\eta \leq (1 + R^{-1})^{-1}$ , so for  $x, y \in \Delta$ ,

$$|\log \tilde{\psi}(x) - \log \psi(y)| \leq |\tilde{\psi}^{-1}|_\infty |\tilde{\psi}(x) - \tilde{\psi}(y)| \leq \frac{R+1}{1+R^{-1}} d_\Delta(x, y) = R d_\Delta(x, y).$$

Thus  $|\tilde{\psi}|_{\eta, \ell} \leq R$ . We have shown that  $\tilde{\psi} \in \mathcal{A}$ , and hence  $\psi \in \mathcal{B}$ . Clearly,  $\psi' \in \mathcal{B}$ .

We have shown that  $\psi$  and  $\psi'$  satisfy the hypotheses of Proposition 4.9 and hence admit the decompositions given in the conclusion of Proposition 4.9. We are therefore in the situation described in Subsection 4.1 (with  $C = \|\phi\|_\eta(1 + R^{-1})$ , and the argument there shows that

$$\int_\Delta |L^{N+n}\phi| dm_\Delta \leq 2\|\phi\|_\eta(1 + R^{-1})\mathbb{P}(h > n).$$

To prove Theorem 2.7, it remains to estimate the decay of  $\mathbb{P}(h > n)$ .

Recall that  $W_k$  is the subset of  $W$  consisting of words of length  $k$ . Then  $\mathbb{P}(W_k) = (1 - p_{-1})^k p_{-1}$ . Elements of  $W_k$  have the form  $w_1 \cdots w_k$  where  $w_1, \dots, w_k$  can be regarded as independent identically distributed random variables, drawn from  $\mathbb{N}$  with distribution  $\mathbb{P}(w_1 = n) = p_n/(1 - p_{-1})$ . Also,  $\mathbb{P}(|w| \geq n) = (1 - p_{-1})^n$ .

### Polynomial tails

**Proposition 4.10** *Suppose that there exists  $C_\tau > 0$  and  $\beta > 1$  such that  $m(\tau \geq n) \leq C_\tau n^{-\beta}$  for  $n \geq 1$ .*

*Then  $\mathbb{P}(h \geq n) \leq C n^{-(\beta-1)}$  for  $n \geq 1$ , where  $C$  depends continuously on  $C_\tau$ ,  $\beta$ ,  $R$ ,  $N$  and  $p_{-1}$ .*

**Proof** Let  $\tilde{t}_n = \bar{\tau} e^R m_\Delta(\bigcup_{\ell=n}^\infty \Delta_\ell)$ . Then

$$p_n = t_n - t_{n+1} \leq \tilde{t}_n - \tilde{t}_{n+1} = \bar{\tau} e^R m_\Delta(\Delta_n) = e^R m(\tau \geq n) \leq e^R C_\tau n^{-\beta}.$$

Using the inequality  $\sum_{j \geq n} j^{-\beta} \leq n^{-\beta} + \int_n^\infty x^{-\beta} dx \leq \beta n^{-(\beta-1)}/(\beta-1)$ , we obtain

$$\mathbb{P}(w_1 \geq n) = (1 - p_{-1})^{-1} \sum_{j \geq n} p_j \leq C_\tau e^R (1 - p_{-1})^{-1} \frac{\beta n^{-(\beta-1)}}{\beta-1} = C_1 n^{-(\beta-1)},$$

where  $C_1 = C_\tau e^R (1 - p_{-1})^{-1} \beta (\beta-1)^{-1}$ . It follows that for  $w \in W$ ,  $k \geq 1$ ,

$$\begin{aligned} \mathbb{P}(\Sigma w \geq n \mid w \in W_k) &= \mathbb{P}(w_1 + \cdots + w_k \geq n) \\ &\leq \sum_{j=1}^k \mathbb{P}(w_j \geq n/k) = k \mathbb{P}(w_1 \geq n/k) \leq C_1 k^\beta n^{-(\beta-1)}. \end{aligned}$$

Hence

$$\begin{aligned}\mathbb{P}(\Sigma w \geq n) &= \sum_{k=1}^{\infty} \mathbb{P}(\Sigma w \geq n \mid w \in W_k) \mathbb{P}(W_k) \\ &\leq C_1 n^{-(\beta-1)} \sum_{k=1}^{\infty} k^{\beta} (1-p_{-1})^k p_{-1} = C'_1 n^{-(\beta-1)},\end{aligned}$$

where  $C'_1 = C_1 p_{-1} \sum_{k=1}^{\infty} k^{\beta} (1-p_{-1})^k$ . Finally,

$$\begin{aligned}\mathbb{P}(h(w) \geq n) &= \mathbb{P}(\Sigma w + N | w| \geq n) \\ &\leq \mathbb{P}(\Sigma w \geq n/2) + \mathbb{P}(|w| \geq n/(2N)) \leq C'_1 2^{\beta-1} n^{-(\beta-1)} + (1-p_{-1})^{n/(2N)}.\end{aligned}$$

The result follows.  $\blacksquare$

### (Stretched) exponential tails

**Proposition 4.11** *Let  $X_1, \dots, X_k$  be nonnegative random variables. Suppose that there exist  $\alpha > 0$ ,  $\gamma \in (0, 1]$ , such that*

$$\mathbb{P}(X_j \geq t \mid X_1 = x_1, \dots, X_{j-1} = x_{j-1}) \leq C e^{-\alpha t^{\gamma}}$$

*for all  $t \geq 0$ ,  $1 \leq j \leq k$  and  $x_1, \dots, x_{j-1} \geq 0$ . Then for all  $\beta \in (0, \alpha/2]$ ,  $t \geq 0$ ,*

$$\mathbb{P}(X_1 + \dots + X_k \geq t) \leq (1 + \beta C_1)^k e^{-\beta t^{\gamma}},$$

*where  $C_1$  depends continuously on  $C$ ,  $\gamma$  and  $\alpha$ .*

**Proof** Note that  $\mathbb{E}(e^{\beta X_1^{\gamma}}) = \int_0^{\infty} \mathbb{P}(e^{\beta X_1^{\gamma}} \geq t) dt = 1 + \int_1^{\infty} \mathbb{P}(e^{\beta X_1^{\gamma}} \geq t) dt$ . Making the substitution  $t = e^{\beta s^{\gamma}}$ , we obtain

$$\mathbb{E}(e^{\beta X_1^{\gamma}}) = 1 + \beta \gamma \int_0^{\infty} s^{\gamma-1} e^{\beta s^{\gamma}} \mathbb{P}(X_1 \geq s) ds \leq 1 + C \beta \gamma \int_0^{\infty} s^{\gamma-1} e^{-(\alpha-\beta)s^{\gamma}} ds \leq 1 + \beta C_1,$$

where  $C_1 = C \gamma \int_0^{\infty} s^{\gamma-1} e^{-\frac{1}{2}\alpha s^{\gamma}} ds$ . Similarly,  $\mathbb{E}(e^{\beta X_j^{\gamma}} \mid X_1, \dots, X_{j-1}) \leq 1 + \beta C_1$ . Hence

$$\begin{aligned}\mathbb{E}(e^{\beta(X_1 + \dots + X_k)^{\gamma}}) &\leq \mathbb{E}(e^{\beta(X_1^{\gamma} + \dots + X_k^{\gamma})}) = \mathbb{E}[\mathbb{E}(e^{\beta(X_1^{\gamma} + \dots + X_k^{\gamma})} \mid X_1, \dots, X_{k-1})] \\ &= \mathbb{E}[(e^{\beta(X_1^{\gamma} + \dots + X_{k-1}^{\gamma})} \mathbb{E}(e^{\beta X_k^{\gamma}} \mid X_1, \dots, X_{k-1})] \\ &\leq (1 + \beta C_1) \mathbb{E}(e^{\beta(X_1^{\gamma} + \dots + X_{k-1}^{\gamma})}) \leq \dots \leq (1 + \beta C_1)^k.\end{aligned}$$

The result follows from Markov's inequality.  $\blacksquare$

**Proposition 4.12** *Suppose that there exist  $C_{\tau}$ ,  $A > 0$ ,  $\gamma \in (0, 1]$  such that  $m(\tau \geq n) \leq C_{\tau} e^{-A n^{\gamma}}$  for  $n \geq 1$ .*

*Then  $\mathbb{P}(h \geq n) \leq C e^{-B n^{\gamma}}$  for all  $n \geq 1$ , where  $C > 0$  and  $B \in (0, A)$  depend continuously on  $C_{\tau}$ ,  $A$ ,  $\gamma$ ,  $R$ ,  $N$  and  $p_{-1}$ .*

**Proof** Following the proof of Proposition 4.10,  $p_n \leq e^R m(\tau \geq n) \leq e^R C_\tau e^{-An^\gamma}$ . Using that  $x^q \leq (2q)^q e^{x/2}$  for all  $x, q > 0$ ,

$$\begin{aligned} \sum_{j \geq n} e^{-Aj^\gamma} &\leq e^{-An^\gamma} + \int_n^\infty e^{-At^\gamma} dt = e^{-An^\gamma} + \gamma^{-1} A^{-1/\gamma} \int_{An^\gamma}^\infty e^{-s} s^{\frac{1}{\gamma}-1} ds \\ &\leq e^{-An^\gamma} + C_{A,\gamma} \int_{An^\gamma}^\infty e^{-s/2} ds \leq 3C_{A,\gamma} e^{-\frac{1}{2}An^\gamma}, \end{aligned}$$

where  $C_{A,\gamma} \geq 1$  is a constant depending continuously on  $A, \gamma$ . Hence

$$\mathbb{P}(w_1 \geq n) = (1 - p_{-1})^{-1} \sum_{j \geq n} p_j \leq 3(1 - p_{-1})^{-1} e^R C_\tau C_{A,\gamma} e^{-\frac{1}{2}An^\gamma}.$$

By Proposition 4.11, for  $B \in (0, \frac{1}{4}A]$ ,

$$\mathbb{P}(\Sigma w \geq n \mid w \in W_k) = \mathbb{P}(w_1 + \dots + w_k \geq n) \leq (1 + BC_1)^k e^{-Bn^\gamma},$$

where  $C_1$  depends continuously on  $C_\tau, A, \gamma, R, p_{-1}$ .

Let  $r = (1 + BC_1)(1 - p_{-1})$  and choose  $B$  small enough that  $r < 1$ . Then

$$\mathbb{P}(\Sigma w \geq n) = \sum_{k=0}^\infty \mathbb{P}(\Sigma w \geq n \mid w \in W_k) \mathbb{P}(W_k) \leq e^{-Bn^\gamma} p_{-1} \sum_{k=0}^\infty r^k = C' e^{-Bn^\gamma},$$

where  $C' = p_{-1}(1 - r)^{-1}$ .

Finally,

$$\begin{aligned} \mathbb{P}(h(w) \geq n) &= \mathbb{P}(\Sigma w + N \mid w| \geq n) \\ &\leq \mathbb{P}(\Sigma w \geq n/2) + \mathbb{P}(|w| \geq n/(2N)) \leq C' e^{-Bn^\gamma/2^\gamma} + (1 - p_{-1})^{n/(2N)}. \end{aligned}$$

The result follows. ■

**Proof of Theorem 2.8** As in the proof of Theorem 2.7, we can write  $\phi = C_0(\psi - \psi')$ , where  $C_0 = \|\phi\|_\eta(1 + R^{-1})$ , and  $\psi, \psi' \in \mathcal{A}$  with  $\int_\Delta \psi dm_\Delta = \int_\Delta \psi' dm_\Delta = 1$ . By Corollary 4.2(b),  $|L^n \psi|_{\eta,\ell}, |L^n \psi'|_{\eta,\ell} \leq R$  and  $|L^n \psi|_\infty, |L^n \psi'|_\infty \leq \bar{\tau} e^R$  for all  $n \geq 0$ .

Next

$$|(L^n \psi)(x) - (L^n \psi)(y)| \leq |L^n \psi|_\infty |\log(L^n \psi)(x) - \log(L^n \psi)(y)|,$$

so  $|L^n \psi|_\eta \leq \bar{\tau} e^R R$ . Similarly,  $|L^n \psi'|_\eta \leq \bar{\tau} e^R R$ . Hence

$$\|L^n \phi\|_\eta \leq C_0(\|\psi\|_\eta + \|\psi'\|_\eta) \leq C_0(2\bar{\tau} e^R + 2\bar{\tau} e^R R) \leq C_1 \|\phi\|_\eta,$$

where  $C_1 = 2\bar{\tau} e^R(1 + R)(1 + R^{-1})$ . Let  $\tilde{\phi} = \sum_{k=0}^{d-1} L^k \phi$ . Then  $\|\tilde{\phi}\|_\eta \leq C_1 d \|\phi\|_\eta$ .

For  $r = 0, \dots, d-1$ , define  $\Delta(r) = \{(y, \ell) \in \Delta : \ell \equiv r \pmod{d}\}$ . Then  $f^d : \Delta(r) \rightarrow \Delta(r)$  is a mixing Young tower with data  $\{I_k/d\}$ ,  $\delta$ , replacing the data  $\{I_k\}$ ,  $\delta$ , for  $\Delta$ .

Note that  $\sum_{k=0}^{d-1} 1_{\Delta(r)} \circ f^k \equiv 1$ . Hence for  $r = 0, \dots, d-1$ ,

$$\int_{\Delta(r)} \tilde{\phi} dm_{\Delta} = \sum_{k=0}^{d-1} \int_{\Delta} 1_{\Delta(r)} L^k \phi dm_{\Delta} = \sum_{k=0}^{d-1} \int_{\Delta} 1_{\Delta(r)} \circ f^k \phi dm_{\Delta} = \int_{\Delta} \phi dm_{\Delta} = 0.$$

Thus for each  $r = 0, \dots, d-1$ , we are in the situation of Theorem 2.7 with  $\Delta$ ,  $f$ ,  $\phi$  replaced by  $\Delta(r)$ ,  $f^d$ ,  $\tilde{\phi}$ . In the case of polynomial tails,

$$\begin{aligned} \int_{\Delta} \left| \sum_{k=0}^{d-1} L^{nd+k} \phi \right| dm_{\Delta} &= \sum_{r=0}^{d-1} \int_{\Delta(r)} \left| \sum_{k=0}^{d-1} L^{nd+k} \phi \right| dm_{\Delta} = \sum_{r=0}^{d-1} \int_{\Delta(r)} |L^{nd} \tilde{\phi}| dm_{\Delta} \\ &\leq C \|\tilde{\phi}\|_{\eta} (nd)^{-(\beta-1)} \leq CC_1 d^{-(\beta-1)} \|\phi\|_{\eta} n^{-(\beta-1)}, \end{aligned}$$

and similarly for the (stretched) exponential case. ■

## 5 Proof for nonuniformly hyperbolic transformations

In this section we prove Theorem 2.10.

The separation time for  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  extends to a separation time on  $\bar{\Delta}$ : define  $s((y, \ell), (y', \ell')) = s(y, y')$  if  $\ell = \ell'$  and 0 otherwise. Recall that we fixed  $\theta \in (0, 1)$ . Define the metric  $d_{\theta}$  on  $\bar{\Delta}$  by setting  $d_{\theta}(p, q) = \theta^{s(p, q)}$ .

Recall that the transfer operator  $P$  corresponding to  $\bar{F} : \bar{Y} \rightarrow \bar{Y}$  and  $\bar{\mu}_Y$  has the form  $(P\phi)(y) = \sum_{a \in \alpha} \zeta(y_a) \phi(y_a)$ . Also,  $(P^n \phi)(y) = \sum_{a \in \alpha_n} \zeta_n(y_a) \phi(y_a)$  where  $\alpha_n = \bigvee_{k=0}^{n-1} \bar{F}^{-k} \alpha$  is the partition of  $\bar{Y}$  into  $n$ -cylinders and  $\zeta_n = \zeta \circ \bar{F} \cdots \zeta \circ \bar{F}^{n-1}$ .

**Proposition 5.1** *Let  $a \in \alpha_n$  and  $y, y' \in a$ . Then (a)  $K_1^{-1} \bar{\mu}_Y(a) \leq \zeta_n(y) \leq K_1 \bar{\mu}_Y(a)$ , (b)  $|\zeta_n(y) - \zeta_n(y')| \leq K_1 \bar{\mu}_Y(a) d_{\theta}(\bar{F}^n y, \bar{F}^n y')$ , where  $K_1 = e^{(1-\theta)^{-1}K} (1-\theta)^{-1}K$ .*

**Proof** It follows from (2.1) that

$$|\log \zeta_n(y) - \log \zeta_n(y')| \leq (1-\theta)^{-1} K d_{\theta}(\bar{F}^n y, \bar{F}^n y').$$

Hence  $\sup_a \zeta_n \leq e^{(1-\theta)^{-1}K} \inf_a \zeta_n$  and

$$\inf_a \zeta_n = \inf P^n 1_a \leq \int_{\bar{Y}} P^n 1_a d\bar{\mu}_Y = \int_{\bar{Y}} 1_a d\bar{\mu}_Y = \bar{\mu}_Y(a).$$

Thus  $\sup_a \zeta_n \leq K_1 \bar{\mu}_Y(a)$ . Similarly,  $\inf_a \zeta_n \geq K_1^{-1} \bar{\mu}_Y(a)$ . Finally,

$$|\zeta_n(y) - \zeta_n(y')| \leq \sup_a \zeta_n |\log \zeta_n(y) - \log \zeta_n(y')| \leq K_1 \bar{\mu}_Y(a) d_{\theta}(\bar{F}^n y, \bar{F}^n y').$$

■

The transfer operator  $L$  corresponding to  $\bar{f} : \bar{\Delta} \rightarrow \bar{\Delta}$  and  $\bar{\mu}_\Delta$  can be written as

$$(L\phi)(p) = \sum_{\bar{f}q=p} g(q)\phi(q), \quad \text{where} \quad g(y, \ell) = \begin{cases} \zeta(y), & \ell = \tau(y) - 1, \\ 1, & \ell < \tau(y) - 1. \end{cases}$$

Then  $(L^n\phi)(p) = \sum_{\bar{f}^n q=p} g_n(q)\phi(q)$  where  $g_n = g \circ g \circ \bar{f} \cdots \circ \bar{f}^{n-1}$ .

Define  $\bar{\Delta}_0 = \{(y, \ell) \in \bar{\Delta} : \ell = 0\}$  and  $\Delta_0 = \{(y, \ell) \in \Delta : \ell = 0\}$ .

**Proposition 5.2** *Let  $p, p' \in \bar{\Delta}$  with  $s(p, p') \geq n \geq 1$ . Then (a)  $\sum_{\bar{f}^n q=p} g_n(q) = 1$ , (b)  $|g_n(p) - g_n(p')| \leq K_1^2 g_n(p) d_\theta(\bar{f}^n p, \bar{f}^n p')$ .*

**Proof** Part (a) is immediate since  $L^n 1 = 1$ . Let  $r(p) = \#\{j \in \{1, \dots, n\} : \bar{f}^j p \in \bar{\Delta}_0\}$ . Note that  $r(p) = r(p')$ . If  $r(p) = 0$ , then  $g_n(p) = g_n(p') = 1$  and (b) holds trivially. Otherwise, we can write  $p = (y, \ell)$ ,  $p' = (y', \ell)$  with  $y, y' \in \bar{Y}$  and  $\ell \geq 0$ . Then  $g_n(p) = \zeta_{r(p)}(y)$  and  $g_n(p') = \zeta_{r(p)}(y')$ .

Let  $a \in \alpha_{r(p)}$  be the cylinder containing  $y$  and  $y'$ . Then by Proposition 5.1,

$$\begin{aligned} |g_n(p) - g_n(p')| &= |\zeta_{r(p)}(y) - \zeta_{r(p)}(y')| \leq K_1 \bar{\mu}_Y(a) d_\theta(\bar{F}^{r(p)} y, \bar{F}^{r(p)} y') \\ &\leq K_1^2 \zeta_{r(p)}(y) d_\theta(\bar{f}^n p, \bar{f}^n p') = K_1^2 g_n(p) d_\theta(\bar{f}^n p, \bar{f}^n p'), \end{aligned}$$

proving (b). ■

**Nonuniform expansion/contraction** Recall that  $\pi : \Delta \rightarrow M$  denotes the projection  $\pi(y, \ell) = T^\ell y$ . For  $p = (x, \ell), q = (y, \ell) \in \Delta$ , we write  $q \in W^s(p)$  if  $y \in W^s(x)$  and  $q \in W^u(p)$  if  $y \in W^u(x)$ . Conditions (P1) translate as follows.

(P2) There exist constants  $K_0 > 0$ ,  $\rho_0 \in (0, 1)$  such that for all  $p, q \in \Delta$ ,  $n \geq 1$ ,

- (i) If  $q \in W^s(p)$ , then  $d(\pi f^n p, \pi f^n q) \leq K_0 \rho_0^{\kappa_n(p)}$ , and
- (ii) If  $q \in W^u(p)$ , then  $d(\pi f^n p, \pi f^n q) \leq K_0 \rho_0^{s(p, q) - \kappa_n(p)}$ ,

where  $\kappa_n(p) = \#\{j = 1, \dots, n : f^j p \in \Delta_0\}$  is the number of returns of  $p$  to  $\Delta_0$  by time  $n$ . It is immediate from conditions (P2) and the product structure on  $Y$  that

$$d(\pi f^n p, \pi f^n q) \leq 2K_0 \rho_0^{\min\{\kappa_n(p), s(p, q) - \kappa_n(p)\}} \quad \text{for all } p, q \in \Delta, n \geq 1. \quad (5.1)$$

**Approximation of observables** Given  $C^\eta$  observables  $v, w : M \rightarrow \mathbb{R}$ , let  $\phi = v \circ \pi$ ,  $\psi = w \circ \pi : \Delta \rightarrow \mathbb{R}$  be the lifted observables. For each  $n \geq 1$ , define  $\tilde{\phi}_n : \Delta \rightarrow \mathbb{R}$ ,

$$\tilde{\phi}_n(p) = \inf\{\phi(f^n q) : s(p, q) \geq 2\kappa_n(p)\}.$$

**Proposition 5.3** *The function  $\tilde{\phi}_n$  lies in  $L^\infty(\Delta)$  and projects down to a Lipschitz observable  $\bar{\phi}_n : \bar{\Delta} \rightarrow \mathbb{R}$ . Moreover, setting  $K_2 = 1 + K_1^2 + 2^\eta K_0^\eta$ ,  $\rho = \rho_0^\eta$  and  $\theta = \rho$ ,*

- (a)  $|\bar{\phi}_n|_\infty = |\tilde{\phi}_n|_\infty \leq |v|_\infty$ , (b)  $|\phi \circ f^n(p) - \tilde{\phi}_n(p)| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{\kappa_n(p)}$  for  $p \in \Delta$ ,  
(c)  $\|L^n \bar{\phi}_n\|_\theta \leq K_2 \|v\|_{C^\eta}$ , for all  $n \geq 1$ .

**Proof** This is standard, see for example [23, Proposition B.5]. We give the details for completeness. If  $s(p, q) \geq 2\kappa_n(p)$ , then  $\tilde{\phi}_n(p) = \tilde{\phi}_n(q)$ . It follows that  $\tilde{\phi}_n$  is piecewise constant on a measurable partition of  $\Delta$ , and hence is measurable, and that  $\bar{\phi}_n$  is well-defined. Part (a) is immediate.

Recall that  $\phi = v \circ \pi$  where  $v : M \rightarrow \mathbb{R}$  is  $C^\eta$ . Let  $p \in \Delta$ . By (5.1) and the definition of  $\tilde{\phi}_n$ ,

$$\begin{aligned} |\phi \circ f^n(p) - \tilde{\phi}_n(p)| &= |v(\pi f^n p) - v(\pi f^n q)| \leq \|v\|_{C^\eta} d(\pi f^n p, \pi f^n q)^\eta \\ &\leq 2^\eta K_0^\eta \rho^{\min\{\kappa_n(p), s(p, q) - \kappa_n(p)\}}, \end{aligned}$$

where  $q$  is such that  $s(p, q) \geq 2\kappa_n(p)$ . In particular,  $s(p, q) - \kappa_n(p) \geq \kappa_n(p)$ , so we obtain part (b).

For part (c), first note that  $|L^n \bar{\phi}_n|_\infty \leq |\bar{\phi}_n|_\infty \leq |v|_\infty$ . Let  $\bar{p} = (y, \ell) \in \bar{\Delta}$  and  $\bar{p}' = (y', \ell') \in \bar{\Delta}$ . If  $d_\theta(\bar{p}, \bar{p}') = 1$ , then

$$|(L^n \bar{\phi})(\bar{p}) - (L^n \bar{\phi})(\bar{p}')| \leq 2|v|_\infty = 2|v|_\infty d_\theta(\bar{p}, \bar{p}').$$

Otherwise, we can write

$$(L^n \bar{\phi}_n)(\bar{p}) - (L^n \bar{\phi}_n)(\bar{p}') = I_1 + I_2,$$

where

$$I_1 = \sum_{\bar{f}^n \bar{q} = \bar{p}} g_n(\bar{q}) (\bar{\phi}_n(\bar{q}) - \bar{\phi}_n(\bar{q}')), \quad I_2 = \sum_{\bar{f}^n \bar{q} = \bar{p}} (g_n(\bar{q}) - g_n(\bar{q}')) \bar{\phi}_n(\bar{q}').$$

As usual, preimages  $\bar{q}, \bar{q}'$  are matched up so that  $s(\bar{q}, \bar{q}') = \kappa_n(\bar{q}) + s(\bar{p}, \bar{p}')$ .

By Proposition 5.2,

$$|I_2| \leq K_1^2 |v|_\infty \sum_{\bar{f}^n \bar{q} = \bar{p}} g_n(\bar{q}) d_\theta(\bar{f}^n \bar{q}, \bar{f}^n \bar{q}') = K_1^2 |v|_\infty d_\theta(\bar{p}, \bar{p}').$$

We claim that  $|\bar{\phi}_n(\bar{q}) - \bar{\phi}_n(\bar{q}')| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{s(\bar{p}, \bar{p}')}.$  Taking  $\theta = \rho$ , it then follows from Proposition 5.2(a) that  $|I_1| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} d_\theta(\bar{p}, \bar{p}')$ .

It remains to verify the claim. Choose  $q, q' \in \Delta$  that project onto  $\bar{q}, \bar{q}' \in \bar{\Delta}$ , so

$$s(q, q') = s(\bar{q}, \bar{q}') = \kappa_n(\bar{q}) + s(\bar{p}, \bar{p}').$$

Write  $\bar{\phi}_n(\bar{q}) - \bar{\phi}_n(\bar{q}') = \phi \circ f^n(\hat{q}) - \phi \circ f^n(\hat{q}')$ , where  $\hat{q}, \hat{q}' \in \Delta$  satisfy

$$s(\hat{q}, q) \geq 2\kappa_n(\bar{q}) \quad \text{and} \quad s(\hat{q}', q') \geq 2\kappa_n(\bar{q}).$$

Since  $\bar{\phi}_n(\bar{q}) = \bar{\phi}_n(\bar{q}')$  if  $s(\bar{q}, \bar{q}') \geq 2\kappa_n(\bar{q})$ , we may suppose without loss that

$$s(\hat{q}, \hat{q}') = s(\bar{q}, \bar{q}') \leq 2\kappa_n(\bar{q}) = 2\kappa_n(\hat{q}).$$

Then

$$s(\bar{p}, \bar{p}') = s(\hat{q}, \hat{q}') - \kappa_n(\hat{q}) \leq \kappa_n(\hat{q}).$$

As in part (b),

$$|\phi \circ f^n(\hat{q}) - \phi \circ f^n(\hat{q}')| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{\min\{\kappa_n(\hat{q}), s(\hat{q}, \hat{q}') - \kappa_n(\hat{q})\}} = 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{s(\bar{p}, \bar{p}')}. \quad \blacksquare$$

This completes the proof of the claim. ■

**Corollary 5.4** *Suppose  $\{b_n\}$ ,  $n \geq 0$  is a nonnegative non-increasing sequence, and  $|L^n \phi|_1 \leq b_n \|\phi\|_\theta$  for all  $n$  and all mean zero  $d_\theta$ -Lipschitz functions  $\phi : \bar{\Delta} \rightarrow \mathbb{R}$ . Then*

$$\left| \int_M v w \circ T^n d\mu - \int_M v d\mu \int_M w d\mu \right| \leq (2^\eta K_0^\eta |\rho^{\kappa_{[n/2]}|_1} + 2K_2 b_{[n/2]}) \|v\|_{C^\eta} \|w\|_{C^\eta}.$$

**Proof** Suppose without loss that  $v$  is mean zero. Since  $\pi : \Delta \rightarrow M$  is a semiconjugacy and  $\mu = \pi_* \mu_\Delta$ , it is equivalent to estimate  $\int_\Delta \phi \psi \circ f^n d\mu_\Delta$ , where  $\phi, \psi : \Delta \rightarrow \mathbb{R}$ ,  $\phi = v \circ \pi$  and  $\psi = w \circ \pi$ . Assume for simplicity that  $n$  is even; the proof for  $n$  odd requires little modification. Let  $\ell \geq 1$ , and write

$$\int_\Delta \phi \psi \circ f^n d\mu_\Delta = \int_\Delta \phi \circ f^\ell \psi \circ f^{\ell+n} d\mu_\Delta = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_\Delta (\phi \circ f^\ell - \tilde{\phi}_\ell) \psi \circ f^{\ell+n} d\mu_\Delta, & I_2 &= \int_\Delta \tilde{\phi}_\ell (\psi \circ f^{n/2} - \tilde{\psi}_{n/2}) \circ f^{\ell+n/2} d\mu_\Delta, \\ I_3 &= \int_\Delta \left( \tilde{\phi}_\ell - \int_\Delta \tilde{\phi}_\ell d\mu_\Delta \right) \tilde{\psi}_{n/2} \circ f^{\ell+n/2} d\mu_\Delta, & I_4 &= \int_\Delta \tilde{\phi}_\ell d\mu_\Delta \int_\Delta \tilde{\psi}_{n/2} d\mu_\Delta. \end{aligned}$$

By Proposition 5.3(b),  $|I_1| \leq |\phi \circ f^\ell - \tilde{\phi}_\ell|_1 |\psi|_\infty \leq 2^\eta K_0^\eta |\rho^{\kappa_\ell}|_1 \|v\|_{C^\eta} |w|_\infty$ . By Proposition 5.3(a,b),  $|I_2| \leq |\tilde{\phi}_\ell|_\infty |\psi \circ f^{n/2} - \tilde{\psi}_{n/2}|_1 \leq 2^\eta K_0^\eta |v|_\infty \|w\|_{C^\eta} |\rho^{\kappa_{n/2}}|_1$ . By Proposition 5.3(c),

$$\begin{aligned} |I_3| &= \left| \int_{\bar{\Delta}} L^{n/2} \left( L^\ell \bar{\phi}_\ell - \int_{\bar{\Delta}} \bar{\phi}_\ell d\bar{\mu}_\Delta \right) \bar{\psi}_{n/2} d\bar{\mu}_\Delta \right| \\ &\leq |L^{n/2} (L^\ell \bar{\phi}_\ell - \int_{\bar{\Delta}} \bar{\phi}_\ell d\bar{\mu}_\Delta)|_1 |\bar{\psi}_{n/2}|_\infty \leq 2b_{n/2} \|L^\ell \bar{\phi}_\ell\|_\theta |w|_\infty \leq 2K_2 b_{n/2} \|v\|_{C^\eta} |w|_\infty. \end{aligned}$$

Finally,  $|I_4| \leq |\int_{\bar{\Delta}} \tilde{\phi}_\ell d\bar{\mu}_\Delta| |w|_\infty = |\int_{\bar{\Delta}} (\tilde{\phi}_\ell - \phi \circ f^\ell) d\bar{\mu}_\Delta| |w|_\infty \leq 2^\eta K_0^\eta \|v\|_{C^\eta} |w|_\infty |\rho^{\kappa_\ell}|_1$  by another application of Proposition 5.3(b).



Altogether,

$$\left| \int_{\Delta} \phi \psi \circ f^n d\mu_{\Delta} \right| \leq (2^{\eta} K_0^{\eta} |\rho^{\kappa_n/2}|_1 + 2K_2 b_{n/2} + 2^{\eta+1} K_0^{\eta} |\rho^{\kappa_{\ell}}|_1) \|v\|_{C^{\eta}} \|w\|_{C^{\eta}}.$$

Letting  $\ell \rightarrow \infty$  yields the result.  $\blacksquare$

By Corollary 5.4 and Theorem 2.7, it remains to estimate  $|\rho^{\kappa_n}|_1$ . A first step towards this is:

**Lemma 5.5**  $\int_{\bar{\Delta}} \rho^{\kappa_n} d\bar{\mu}_{\Delta} \leq 2\bar{\tau}^{-1} \sum_{j>n/3} \bar{\mu}_Y(\tau \geq j) + n \sum_{k=0}^{\infty} \rho^{k+1} \bar{\mu}_Y(\tau_k \geq n/3)$ , where  $\tau_k = \sum_{j=0}^{k-1} \tau \circ \bar{F}^k$ .

**Proof** First write  $\int_{\bar{\Delta}} \rho^{\kappa_n} d\bar{\mu}_{\Delta} = \sum_{k=0}^{\infty} \rho^k \bar{\mu}_{\Delta}(\kappa_n = k)$ . Note that  $\kappa_n(p) = 0$  if and only if  $f^j(p) \notin \bar{\Delta}_0$  for all  $j = 1, \dots, n$ , so  $\bar{\mu}_{\Delta}(\kappa_n = 0) = \bar{\tau}^{-1} \sum_{j \geq n} \bar{\mu}_Y(\tau > j)$ .

When  $\kappa_n(p) \geq 1$ , we can define  $r(p) = \min\{j \in \{1, \dots, n\} : f^j p \in \bar{\Delta}_0\}$  and  $s(p) = \max\{j \in \{1, \dots, n\} : f^j p \in \bar{\Delta}_0\}$ . Hence for  $k \geq 1$ ,

$$\{\kappa_n(p) = k\} = \bigcup_{1 \leq r \leq s \leq n} \{\kappa_n(p) = k, r(p) = r, s(p) = s\}.$$

It is easy to check that  $\bar{\mu}_{\Delta}\{r(p) = j\} = \bar{\tau}^{-1} \bar{\mu}_Y(\tau \geq j)$ , so

$$\bar{\mu}_{\Delta}(\kappa_n(p) = k) \leq \bar{\tau}^{-1} \sum_{j>n/3} \bar{\mu}_Y(\tau \geq j) + b_{n,k},$$

where

$$\begin{aligned} b_{n,k} &= \sum_{0 \leq r \leq n/3} \sum_{2n/3 \leq s \leq n} \bar{\mu}_{\Delta}(\kappa_n(p) = k, r(p) = r, s(p) = s) \\ &= \sum_{0 \leq r \leq n/3} \sum_{2n/3 \leq s \leq n} \bar{\mu}_{\Delta}(\kappa_{s-r}(f^r p) = k-1, r(p) = r, s(p) = s) \\ &\leq \sum_{0 \leq r \leq n/3} \sum_{2n/3 \leq s \leq n} \bar{\mu}_{\Delta}(\kappa_{s-r}(f^r p) = k-1, f^r p \in \bar{\Delta}_0, f^s p \in \bar{\Delta}_0) \\ &= \sum_{0 \leq r \leq n/3} \sum_{2n/3 \leq s \leq n} \bar{\mu}_{\Delta}(\kappa_{s-r}(p) = k-1, p \in \bar{\Delta}_0, f^{s-r} p \in \bar{\Delta}_0) \\ &\leq n \sum_{j \geq n/3} \bar{\mu}_{\Delta}(\kappa_j(p) = k-1, p \in \bar{\Delta}_0, f^j p \in \bar{\Delta}_0) \\ &= n\bar{\tau}^{-1} \sum_{j \geq n/3} \bar{\mu}_Y(y \in \bar{Y}, f^j y \in \bar{Y}, \tau_{k-1}(y) = j) \leq n\bar{\tau}^{-1} \bar{\mu}_Y(y \in \bar{Y} : \tau_{k-1}(y) \geq n/3). \end{aligned}$$

This completes the proof.  $\blacksquare$

**Proof of Theorem 2.10** We restrict from now on to the cases of polynomial tails and stretched exponential tails. The sum  $\sum_{j \geq n} \bar{\mu}_Y(\tau > j)$  is estimated in the same way as  $\mathbb{P}(w_1 \geq n)$  in the proofs of Propositions 4.10 and 4.11, so it remains to show that  $n \sum_{k=0}^{\infty} \rho^k \bar{\mu}_Y(\tau_k \geq n)$  satisfies the required estimate.

In the case of polynomial tails,  $\bar{\mu}_Y(\tau_k \geq n) \leq k \bar{\mu}_Y(\tau \geq n/k) \leq C_\tau k^{\beta+1} n^{-\beta}$ , and so  $n \sum_{k=1}^{\infty} \rho^k \bar{\mu}_Y(\tau_k \geq n) \leq C_2 n^{-(\beta-1)}$  where  $C_2 = C_\tau \sum_{k=1}^{\infty} \rho^k k^{\beta+1}$ .

It remains to treat the stretched exponential case. Writing  $X_j = \tau \circ F_j$ ,

$$\begin{aligned} \bar{\mu}_Y(X_0 = j_0, \dots, X_k = j_k) &= \int_{\bar{Y}} 1_{\{\tau \circ F^k = j_k\}} 1_{\{X_0 = j_0, \dots, X_{k-1} = j_{k-1}\}} d\bar{\mu}_Y \\ &= \int_{\bar{Y}} 1_{\{\tau = j_k\}} P^k 1_{\{X_0 = j_0, \dots, X_{k-1} = j_{k-1}\}} d\bar{\mu}_Y \\ &\leq \bar{\mu}_Y(\tau = j_k) |P^k 1_{\{X_0 = j_0, \dots, X_{k-1} = j_{k-1}\}}|_{\infty} \end{aligned}$$

By Proposition 5.1(a),

$$\begin{aligned} (P^k 1_{\{X_0 = j_0, \dots, X_{k-1} = j_{k-1}\}})(y) &= \sum_{a \in \alpha_k} \zeta_k(y_a) 1_{\{X_0 = j_0, \dots, X_{k-1} = j_{k-1}\}} \\ &\leq K_1 \sum_{a \in \alpha_k} \bar{\mu}_Y(a) 1_{\{\tau(a) = j_0, \dots, \tau(F^{k-1}a) = j_{k-1}\}} \\ &= K_1 \bar{\mu}_Y(\tau = j_0, \dots, \tau \circ F^{k-1} = j_{k-1}). \end{aligned}$$

Hence

$$\bar{\mu}_Y(X_0 = j_0, \dots, X_k = j_k) \leq K_1 \bar{\mu}_Y(\tau = j_k) \bar{\mu}_Y(X_0 = j_0, \dots, X_{k-1} = j_{k-1}),$$

and so

$$\bar{\mu}_Y(X_k \geq n \mid X_0 = j_0, \dots, X_{k-1} = j_{k-1}) \leq K_1 \bar{\mu}_Y(\tau \geq n) \leq K_1 C_\tau e^{-An^\gamma}.$$

By Proposition 4.11, there exists  $B \in (0, A)$  and  $C_B \in (0, \rho)$  depending continuously on  $C_\tau$ ,  $\gamma$  and  $A$  such that

$$\mu_Y(\tau_k \geq n) = \mu_Y(X_0 + \dots + X_{k-1} \geq n) \leq C_B^k e^{-Bn^\gamma},$$

Hence  $\sum_{k=1}^{\infty} \rho^k \mu_Y(\tau_k \geq n) \leq \{\sum_{k=1}^{\infty} (\rho C_B)^k\} e^{-Bn^\gamma}$  as required.  $\blacksquare$

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